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Stability Boundary Characterization of Nonlinear Autonomous Dynamical Systems in the Presence of a Type-Zero Saddle-Node Equilibrium Point[1]

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Abstract. Under the assumption that all equilibrium points are hyperbolic, the stability boundary of nonlinear autonomous dynamical systems is characterized as the union of the stable manifolds of equilibrium points on the stability boundary. The existing characterization of the stability boundary is extended in this paper to consider the existence of non-hyperbolic equilibrium points on the stability boundary. In particular, a complete characterization of the stability boundary is presented when the system possesses a type-zero saddle-node equilibrium point on the stability boundary. It is shown that the stability boundary consists of the stable manifolds of all hyperbolic equilibrium points on the stability boundary and of the stable manifold of the type-zero saddle-node equilibrium point.

Keywords. Stability Region, Stability Boundary, Saddle-node Bifurcation.

1. Introduction

Usually, asymptotically stable equilibrium of autonomous dynamical systems are not globally stable. In fact, there is a subset of the state space called stability region (basin of attraction) composed of all initial conditions whose trajectories converge to the asymptotically stable equilibrium point as time tends to infinity.

The problem of determining stability regions of nonlinear dynamical systems is of fundamental importance for many applications in engineering and sciences [1], [3], [10]. For example, estimates of the stability region are used in power systems to estimate the maximal time the breaker can trip a transmission line after the ocurrence of a short-circuit without causing instability [3]. The size of the stability

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region has also great importance in control theory. It has been shown that the stability region of closed-loop nonlinear systems shrinks as the feedback control gain is increased, leading to destabilization [9].

Optimal estimates of the stability region can be obtained exploring the characterization of the stability boundary (the boundary of the stability region) [4]. Comprehensive characterizations of the stability boundary of classes of nonlinear dynamical systems can be found, for example, in [2]. The existing characterizations of stability boundaries are proved under the key assumption that all the equilibrium points on the stability boundary are hyperbolic. In this paper, however, we are interested in studing of the stability boundary when the system is subject to parameter variation. Under parameter variation, local bifurcations may occur on the stability boundary and the assumption of hyperbolicity of equilibrium points may be violated. The characterization of the stability boundary in the presence of non-hyperbolic equilibrium points is of fundamental importance to understand how the stability region behaves under parameter variation.

In this paper, we study the stability boundary characterization in the presence of a type-zero saddle-node equilibrium point. Necessary and sufficient conditions for a type-zero saddle-node equilibrium point lying on the stability boundary are presented. A complete characterization of the stability boundary when the system possesses a type-zero saddle-node equilibrium point on the stability boundary is also presented. It is shown that the stability boundary consists of the stable manifolds of all hyperbolic equilibrium points on the stability boundary union with the stable manifold of the type-zero saddle-node equilibrium point on the stability boundary.

2. Preliminaries on Dynamical Systems

In this section, some classical concepts of the theory of dynamical systems are reviewed. More details on the content explored in this section can be found in [5, 12].

Consider the nonlinear autonomous dynamical system

$$\dot{x} = f(x) \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field of class C^r with $r \geq 1$. The solution of (2.1) starting at x at time $t = 0$ is denoted by $\varphi(t, x)$.

A point $x^* \in \mathbb{R}^n$ is an equilibrium point of (2.1) if $f(x^*) = 0$. An equilibrium point x^* of (2.1) is said to be *hyperbolic* if none of the eigenvalues of the Jacobian matrix $D_x f(x^*)$ has real part equal to zero. Moreover, a hyperbolic equilibrium point x^* is of *type* k if the Jacobian matrix possesses k eigenvalues with positive real part and $n - k$ eigenvalues with negative real part. A set $S \in \mathbb{R}^n$ is said to be an *invariant set* of (2.1) if every trajectory of (2.1) starting in S remains in S for all t.

Given an equilibrium point x^* of the nonlinear autonomous dynamical system (2.1), the space $\mathbb{R}^n$ can be decomposed as a direct sum of three subspaces denoted by $E^s = span\ \{e_1, ..., e_s\}$, the stable subspace, $E^u = span\ \{e_{s+1}, ..., e_{s+u}\}$, the unstable subspace and $E^c = span\ \{e_{s+u+1}, ..., e_{s+u+c}\}$, the center subspace, with

$s+u+c = n$, which are invariant with respect to the linearized system $\dot{\xi} = D_x f(x^*)\xi$. The generalized eigenvectors $\{e_1, ..., e_s\}$ of the jacobian matrix associated with the eigenvalues that have negative real part span the stable subspace E^s, whereas the generalized eigenvectors $\{e_{s+1}, ..., e_{s+u}\}$ and $\{e_{s+u+1}, ..., e_{s+u+c}\}$, respectively associated with the eigenvalues that have positive and zero real part, span the unstable and center subspaces.

If x^* is an equilibrium point of (2.1), then there exist invariant local manifolds $W^s_{loc}(x^*)$, $W^{cs}_{loc}(x^*)$, $W^c_{loc}(x^*)$, $W^u_{loc}(x^*)$ and $W^{cu}_{loc}(x^*)$ of class C^r, tangent to E^s, $E^c \oplus E^s$, E^c, E^u and $E^c \oplus E^u$ at x^*, respectively [5, 7]. These manifolds are respectively called stable, stable center, center, unstable and unstable center manifolds. The stable and unstable manifolds are unique, but the stable center, center and unstable center manifolds may not be.

The system $\dot{x} = f(x)$ is called topologically equivalent to a dynamical system $\dot{x} = g(x)$ if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

The idea of transversality is basic in the study of dynamical systems. The transversal intersection is notorious because it persists under perturbations of the vector field [6]. The manifolds M and N of class C^r, with $r \geq 1$, in $\mathbb{R}^n$, satisfy the *transversality condition* if either (i) the tangent spaces of M and N span the tangent space of $\mathbb{R}^n$ at every point x of the intersection $M \cap N$,

$$i.e.,\ T_x(M) + T_x(N) = T_x(\mathbb{R}^n) \quad for\ all\ x \in M \cap N$$

or (ii) they do not intersect at all.

3. Type-Zero Saddle-Node Equilibrium Point

In this section, a specific type of non-hyperbolic equilibrium point, namely type-zero saddle-node equilibrium point, is studied. In particular, the dynamical behavior in a neighborhood of the equilibrium is investigated in details including the asymptotic behavior of solutions in the invariant local manifolds.

Consider the nonlinear dynamical system (2.1).

Definition 3.1 ([5])**.** *A non-hyperbolic equilibrium point $p \in \mathbb{R}^n$ of (2.1), is called a saddle-node equilibrium point if the following conditions are satisfied:*
(i) $D_x f(p)$ has a unique simple eigenvalue 0 with right eigenvector v and left eigenvector w.
(ii) $w(D^2_x f(p)(v, v)) \neq 0$.

Saddle-node equilibrium points can be classified in types, according to the number of eigenvalues of $D_x f(p)$ with positive real part.

Definition 3.2. *A saddle-node equilibrium point p of (2.1), is called a type-k saddle-node equilibrium point if $D_x f(p)$ has k eigenvalues with positive real part and $n - k - 1$ with negative real part.*

In this paper, we will be mainly interested in type-zero saddle-node equilibrium points. If p is a type-zero saddle-node equilibrium point, then the following properties hold [13]:

(1) The unidimensional local center manifold $W^c_{loc}(p)$ of p can be splitted in three invariant submanifolds:

$$W^c_{loc}(p) = W^{c^-}_{loc}(p) \cup \{p\} \cup W^{c^+}_{loc}(p)$$

where $q \in W^{c^-}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$ and $q \in W^{c^+}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$.

(2) The $(n-1)$-dimensional local stable manifold $W^s_{loc}(p)$ of p exists, is unique, and if $q \in W^s_{loc}(p)$ then $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$.

(3) There is a neighborhood N of p where the phase portrait of system (2.1) on N is topologically equivalent to the phase portrait of Figure 1.

The stable and unstable manifolds of a hyperbolic equilibrium point are defined extending the local manifolds through the flow [14]. Usually this technique to define the global manifolds can not be applied to non-hyperbolic equilibrium points. However, in the particular case of a type-zero saddle-node non-hyperbolic equilibrium point p, one still can define the global stable manifold $W^s(p)$ and the global center manifold $W^c(p)$ extending the local manifold $W^s_{loc}(p)$ and $W^c_{loc}(p)$ through the flow as follows:

$$W^s(p) := \bigcup_{t \leq 0} \varphi(t, W^s_{loc}(p))$$

$$W^c(p) := W^{c^-}(p) \bigcup \{p\} \bigcup W^{c^+}(p)$$

where

$$W^{c^-}(p) := \bigcup_{t \leq 0} \varphi(t, W^{c^-}_{loc}(p) \quad \text{and} \quad W^{c^+}(p) := \bigcup_{t \geq 0} \varphi(t, W^{c^+}_{loc}(p)).$$

This extension is justified by the invariance and the asymptotic behavior of local stable manifold $W^s_{loc}(p)$ and of the local center manifold $W^c_{loc}(p)$ given by item (1) and (2) above.

Obviously, $q \in W^s(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$, $q \in W^{c^-}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$, and $q \in W^{c^+}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$.

In order to obtain more insights into the dynamical behavior of (2.1), in the neighborhood of a type-zero saddle-node equilibrium point p, a neighborhood $U \subseteq N$ of p will be decomposed into subsets U^+ and U^-. We define

$$U^- := \{q \in U \ : \ \varphi(t,q) \to p \text{ as } t \to \infty\} \quad \text{and} \quad U^+ := U - U^-.$$

For any neighborhood $U \subseteq N$ of p, we obviously have $U = U^- \cup U^+$.

4. Stability Boundary Characterization

In this section, an overview of the existing body theory about the stability boundary characterization of nonlinear dynamical systems is presented.

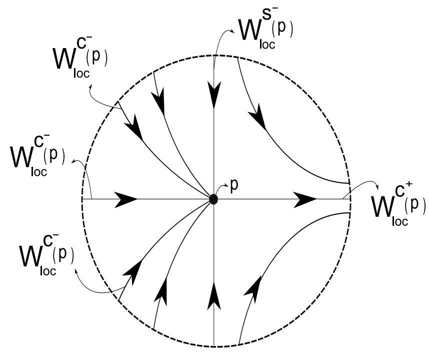

Figure 1: The local manifolds $W^{c^+}_{loc}(p)$ and $W^{s}_{loc}(p)$ are unique, whereas there are infinite choices for $W^{c^-}_{loc}(p)$. Three possible choices for $W^{c^-}_{loc}(p)$ are indicated in this figure.

Suppose x^s is an asymptoticaly stable equilibrium point of (2.1). The *stability region* (or basin of attraction) of x^s is the set

$$A(x^s) = \{x \in \mathbb{R}^n \ : \ \varphi(t,x) \to x^s \text{ as } t \to \infty\},$$

of all initial conditions $x \in \mathbb{R}^n$ whose trajectories converge to x^s when t tends to infinity. The stability region $A(x^s)$ is an open and invariant set. Its closure $\overline{A(x^s)}$ is invariant and the *stability boundary* $\partial A(x^s)$ is a closed and invariant set. If $A(x^s)$ is not dense in $\mathbb{R}^n$, then $\partial A(x^s)$ is of dimension $n-1$ [8].

The unstable equilibrium points that lie on the stability boundary $\partial A(x^s)$ play an essential role in the stability boundary characterization.

Let x^s be a hyperbolic asymptotically stable equilibrium point of (2.1) and consider the following assumptions:

(**A1**) All the equilibrium points on $\partial A(x^s)$ are hyperbolic.

(**A2**) The stable and unstable manifolds of equilibrium points on $\partial A(x^s)$ satisfy the transversality condition.

(**A3**) Trajectories on $\partial A(x^s)$ approaches one of the equilibrium points as $t \to \infty$.

Assumptions (A1) and (A2) are generic properties of dynamical systems in the form of (2.1). In other words, they are satisfied for almost all dynamical systems in the form of (1) and, in practice, do not need to be verified. On the contrary, assumption (A3) is not a generic property of dynamical systems and needs to be checked. The existence of an energy function is a sufficient condition to guarantee the satisfaction of (A3) [2].

Next theorem provides necessary and sufficient conditions to guarantee that an equilibrium point lies on the stability boundary in terms of properties of its stable and unstable manifolds.

Theorem 4.1 ([2]). *Let x^s be a hyperbolic asymptotically stable equilibrium point of (2.1) and $A(x^s)$ be its stability region. If x^* is an equilibrium point of (2.1) and assumptions (A1)-(A3) are satisfied, then:*
(i) $x^* \in \partial A(x^s)$ if and only if $W^u(x^*) \cap A(x^s) \neq \emptyset$.
(ii) $x^* \in \partial A(x^s)$ if and only if $W^s(x^*) \subseteq \partial A(x^s)$.

Exploring Theorem 4.1, next theorem provides a complete characterization of the stability boundary $\partial A(x^s)$. It asserts that the stability boundary $\partial A(x^s)$ is the union of the stable manifolds of the equilibrium points on $\partial A(x^s)$.

Theorem 4.2 ([2])**.** *Let x^s be a hyperbolic asymptotically stable equilibrium point of (2.1) and $A(x^s)$ be its stability region. If assumptions (A1)-(A3) are satisfied, then:*

$$\partial A(x^s) = \bigcup_i W^s(x^i)$$

where x^i, $i = 1, 2, ...$ are the equilibrium points on $\partial A(x^s)$.

Theorem 4.2 provides a complete stability boundary characterization of system (2.1) under assumptions (A1)-(A3). In this paper, we study the characterization of the stability boundary when assumption (A1) is violated. In particular, we study the stability boundary characterization when a type-zero saddle-node non-hyperbolic equilibrium point lies on the stability boundary.

5. Saddle-Node Equilibrium Point on the Stability Boundary

In this section, a complete characterization of the stability boundary in the presence of a type-zero saddle-node equilibrium point is developed.

Next theorem offers necessary and sufficient conditions to guarantee that a type-zero saddle-node equilibrium point lies on the stability boundary in terms of the properties of its stable and center manifolds. They also provide insights into how to develop a computational procedure to check if a type-zero saddle-node equilibrium point lies on the stability boundary.

Theorem 5.1 (Type-zero saddle-node equilibrium point on the stability boundary)**.** *Let p be a type-zero saddle-node equilibrium point of (2.1). Suppose also, the existence of an asymptotically stable equilibrium point x^s and let $A(x^s)$ be its stability region. Then the following holds:*
(i) $p \in \partial A(x^s)$ if and only if $W^{c^+}(p) \cap \overline{A(x^s)} \neq \emptyset$.
(ii) $p \in \partial A(x^s)$ if and only if $(W^s(p) - \{p\}) \cap \partial A(x^s) \neq \emptyset$.

Proof. (i) ($\Longleftarrow$) Suppose that $W^{c^+}(p) \cap \overline{A(x^s)} \neq \emptyset$. Then there exists $x \in W^{c^+}(p) \cap \overline{A(x^s)}$. Note that $\varphi(t, x) \longrightarrow p$ as $t \longrightarrow -\infty$. On the other hand, set $\overline{A(x^s)}$ is invariant, thus $\varphi(t, x) \in \overline{A(x^s)}$ for all $t \leq 0$. As a consequence, $p \in \overline{A(x^s)}$. Since $p \notin A(x^s)$, we have that $p \in \mathbb{R}^n - A(x^s)$. Therefore, $p \in \partial A(x^s)$.
(i) ($\Longrightarrow$) Suppose that $p \in \partial A(x^s)$. Let $B(q, \epsilon)$ be a ball of radius ϵ centered at q for some $q \in W^{c^+}(p)$ and $\epsilon > 0$. Consider a disk D of dimension $n - 1$ contained in $B(q, \epsilon)$ and transversal to $W^{c^+}(p)$ at q. As a consequence of λ-lemma for non-hyperbolic equilibrium points [11], we can affirm that $\cup_{t \leq 0} \varphi(t, B(q, \epsilon)) \supset U^+$ where U is a neighborhood of p. Since $p \in \partial A(x^s)$, we have that $U \cap A(x^s) \neq \emptyset$. On the other hand, $U^- \cap A(x^s) = \emptyset$, thus $U^+ \cap A(x^s) \neq \emptyset$. Thus, there exists a point $p \in B(q, \epsilon)$ and a time t^* such that $\varphi(t^*, p) \in A(x^s)$. Since $A(x^s)$ is invariant, we

have that $p \in A(x^s)$. Since ϵ can be chosen arbitrarily small, we can find a sequence of points $\{p_i\}$ with $p_i \in A(x^s)$ for all $i = 1, 2, ...$ such that $p_i \longrightarrow q$ as $i \longrightarrow \infty$, that is, $q \in \overline{A(x^s)}$. Since $q \in W^{c^+}(p)$, we have that $W^{c^+}(p) \cap \overline{A(x^s)} \neq \emptyset$.
The proof of (ii) is similar to the proof of (i) and will be omitted. □

With some additional assumptions a sharper result regarding type-zero saddle-node equilibrium points on the stable boundary is obtained.

Let x^s be an asymptotically stable equilibrium point, p be a type-zero saddle-node equilibrium point of (2.1), and consider the following assumptions:
(**A1**$'$) All the equilibrium points on $\partial A(x^s)$ are hyperbolic, except possibly for p.
(**A4**) The stable manifold of the equilibrium points on $\partial A(x^s)$ and the manifold $W^{c^+}(p)$ satisfy the transversality condition.

Under assumptions $(A1')$, $(A3)$ and $(A4)$, next theorem offers necessary and sufficient conditions which are sharper than conditions of Theorem 5.1, to guarantee that a type-zero saddle-node equilibrium point lies on the stability boundary of nonlinear autonomous dynamical systems.

Theorem 5.2 (Further characterization of the type-zero saddle-node equilibrium point on the stability boundary)**.** *Let p be a type-zero saddle-node equilibrium point of (2.1). Suppose also, the existence of an asymptotically stable equilibrium point x^s and let $A(x^s)$ be its stability region. If assumptions $(A1')$, $(A3)$ and $(A4)$ are satisfied, then*
(i) $p \in \partial A(x^s)$ if and only if $W^{c^+}(p) \cap A(x^s) \neq \emptyset$.
(ii) $p \in \partial A(x^s)$ if and only if $W^s(p) \subset \partial A(x^s)$.

Proof. (i) $(\Longleftarrow)$ Suppose that $W^{c^+}(p) \cap A(x^s) \neq \emptyset$. Since $W^{c^+}(p) \cap A(x^s) \subset W^{c^+}(p) \cap \overline{A(x^s)}$ we have that $W^{c^+}(p) \cap \overline{A(x^s)} \neq \emptyset$. Thus, from Theorem 5.1, one concludes that $p \in \partial A(x^s)$.
$(i)(\Longrightarrow)$ Suppose that $p \in \partial A(x^s)$. From Theorem 5.1, we can affirm that $W^{c^+}(p) \cap \overline{A(x^s)} \neq \emptyset$. We are going to show, under assumptions $(A1')$, $(A3)$ and $(A4)$, that $W^{c^+}(p) \cap \overline{A(x^s)} \neq \emptyset$ implies $W^{c^+}(p) \cap A(x^s) \neq \emptyset$. Let $q \in W^{c^+}(p) \cap \overline{A(x^s)}$. If $q \in A(x^s)$, then there is nothing to be proved. Suppose that $q \in \partial A(x^s)$. Assumption $(A3)$ asserts the existence of an equilibrium point $x^* \in \partial A(x^s)$ such that $\varphi(t, p) \longrightarrow x^*$ as $t \longrightarrow \infty$. Since $W^{c^-}(p) \cap \partial A(x^s) = \emptyset$ and $W^s(p) \cap W^{c^+}(p) = \emptyset$, we can affirm that $x^* \neq p$. As a consequence of $(A1')$, x^* is a hyperbolic equilibrium point. Since $q \in W^{c^+}(p) \cap W^s(x^*)$, assumption $(A4)$ implies that x^* is a type-zero hyperbolic equilibrium point. But this fact leads us to an absurd, since $x^* \in \partial A(x^s)$. Therefore, $W^{c^+}(p) \cap A(x^s) \neq \emptyset$.
(ii) $(\Longleftarrow)$ Suppose that $W^s(p) \subset \partial A(x^s)$. Since $p \in W^s(p)$, then $p \in \partial A(x^s)$.
$(ii)(\Longrightarrow)$ Suppose that $p \in \partial A(x^s)$. From (i), we have that $W^{c^+}(p) \cap A(x^s) \neq \emptyset$. Let $w \in W^{c^+}(p) \cap A(x^s)$ and $B(w, \epsilon)$ be an open ball with an arbitrarily small radius ϵ centered at w. Radius ϵ can be chosen arbitrarily small such that $B(w, \epsilon) \subset A_{\lambda_0}(x^s_{\lambda_0})$. Let q be an arbitrary point of $W^s(p)$ and consider a disk D that is transversal to $W^s(p)$ at q and is unidimensional. As a consequence of λ-lemma for non-hyperbolic equilibrium points [11], there exists an element $z \in D$ and a time $t^* > 0$ such that $\varphi(t^*, z) \in B(w, \epsilon)$. Since $A(x^s)$ is invariant, we have that

$z \in A(x^s)$. Since ϵ and the disk D can be chosen arbitrarily small, then there exist points of $A(x^s)$ arbitrarily close to q. Therefore $q \in \overline{A(x^s)}$. Since $W^s(p)$ cannot contain points on $A(x^s)$, $q \in \partial A(x^s)$. Exploring the fact that q was arbitrarily taken in $W^s(p)$, we can affirm that $W^s(p) \subset \partial A(x^s)$.

□

Under assumptions $(A1^{'})$, $(A2)-(A4)$, and exploring the results of Theorems 4.1 and 5.2, we obtain the next corollary whose proof is analogous to the proof of Theorem 5.2.

Corollary 5.2 (Hyperbolic equilibrium points on the stability boundary)**.** *Let p be a type-zero saddle-node equilibrium point of (2.1). Suppose also, the existence of a hyperbolic equilibrium point x^* and an asymptotically stable equilibrium point x^s, and let $A(x^s)$ be the stability region of the latter. If assumptions $(A1^{'})$, $(A2)-(A4)$ are satisfied, then*
(i) $x^* \in \partial A(x^s)$ if and only if $W^u(x^*) \cap A(x^s) \neq \emptyset$.
(ii) $x^* \in \partial A(x^s)$ if and only if $W^s(x^*) \subset \partial A(x^s)$.

Corollary 5.2 is a more general result than Theorem 4.2, since assumption (A1) used in the proof of Theorem 4.2 is relaxed.

Exploring the results of Corollary 5.2 and Theorem 5.2, the next theorem provides a complete characterization of the stability boundary when a type-zero saddle-node equilibrium point lies on $\partial A(x^s)$.

Theorem 5.3 (Stability Boundary Characterization)**.** *Let x^s be an asymptotically stable equilibrium point of (2.1) and $A(x^s)$ be its stability region. Suppose also, the existence of a type-zero saddle-node equilibrium point p on the stability boundary $\partial A(x^s)$. If assumptions $(A1^{'})$, $(A2)-(A4)$ are satisfied, then*

$$\partial A(x^s) = \bigcup_i W^s(x^i) \bigcup W^s(p)$$

where x^i, $i = 1, 2, \ldots$ are the hyperbolic equilibrium points on $\partial A(x^s)$.

Proof. If the hyperbolic equilibrium point $x^i \in \partial A(x^s)$, then, from Corollary 5.2, we have that $W^s(x^i) \subset \partial A(x^s)$. Since $p \in \partial A(x^s)$, we have that $W^s(p) \subset \partial A(x^s)$ from Theorem 5.2. Therefore, $\cup_i W^s(x^i) \cup W^s(p) \subset \partial A_{\lambda_0}(x^s_{\lambda_0})$. On the other hand, from assumption $(A3)$, if $q \in \partial A(x^s)$, then we have that $\varphi(t,q) \longrightarrow x^i$ for some i or $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow \infty$. Since the intersection $W^{c^-}(p) \cap \partial A(x^s)$ is empty, we can affirm that $q \in W^s(x^i)$ or $q \in W^s(p)$. Therefore, $\partial A(x^s) \subset \cup_i W^s(x^i) \cup W^s(p)$, and the theorem is proven. □

6. Example

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= x^2 + y^2 - 1 \\ \dot{y} &= x^2 - y - 1 \end{aligned} \tag{6.1}$$

with $(x, y) \in \mathbb{R}^2$.

System (6.1) possesses, three equilibrium points; they are $p = (0, -1)$, a type-zero saddle-node equilibrium point, $x^s = (-1, 0)$, an asymptoticaly stable equilibrium point and $x^* = (1, 0)$, a type-one hyperbolic equilibrium point. Both the type-zero saddle-node equilibrium point and type-one hyperbolic equilibrium point belong to the stability boundary of $x^s = (-1, 0)$. The stability boundary $\partial A(-1, 0)$ is formed, according to Theorem 5.3, as the union of the stable manifold of the type-one hyperbolic equilibrium point $(1, 0)$ and the stable manifold of the type-zero saddle-node equilibrium point $(0, -1)$. See Figure 2.

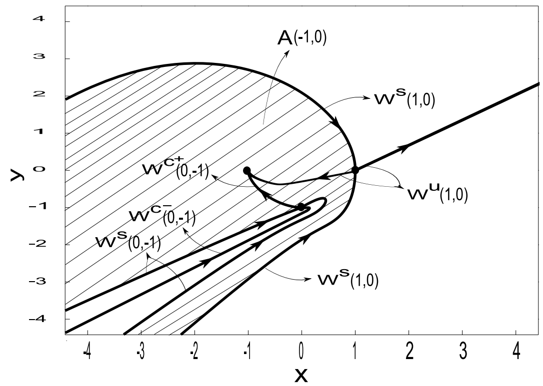

Figure 2: The phase portait of system (6.1). The stability boundary of the asymptotically stable equilibrium point $(-1, 0)$ is composed of the stable manifold of the type-one hyperbolic equilibrium point $(1, 0)$ union with the stable manifold of the type-zero saddle-node equilibrium point $(0, -1)$.

7. Conclusions

Necessary and sufficient conditions for a type-zero saddle-node equilibrium point lying on the stability boundary were presented in this paper. These conditions provide insights into how to develop a computational procedure to check if a type-zero saddle-node equilibrium point lies on the stability boundary. A complete characterization of the stability boundary when the system possesses a type-zero saddle-node equilibrium point on the stability boundary was developed for a class of nonlinear autonomous dynamical systems. This characterization is an important step to study the behavior of the stability boundary under parameter variation.

Resumo. Sob a suposição que todos pontos de equilíbrio são hiperbólico, a fronteira da região de estabilidade de sistemas dinâmicos autônomos não lineares é caracterizada como a união das variedades estáveis dos pontos de equilíbrio na fronteira da região de estabilidade. A caracterização existente da fronteira da região de estabilidade é estendida neste artigo ao considerar a existência de pontos de equilíbrio não hiperbólicos na fronteira da região de estabilidade. Em particular, uma caracterização completa da fronteira da região de estabilidade é apresentada quando o sistema possui um ponto de equilíbrio sela-nó do tipo-zero na fronteira da região de estabilidade. É mostrado que a fronteira da região de establidade consiste das variedades estáveis de todos os pontos de equilíbrio hiperbólico na fronteira da região de estabilidade e da variedade estável do ponto de equilíbrio sela-nó do tipo zero.

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