Tema
Tendências em Matemática Aplicada e Computacional, **18**, N. 2 (2017), 225-232

www.scielo.br/tema
doi: 10.5540/tema.2017.018.02.0225

New Extension for Sub Equation Method and its Application to the Time-fractional Burgers Equation by using of Fractional Derivative

A. NEIRAMEH

Received on August 20, 2016 / Accepted on February 17, 2017

ABSTRACT. In this paper, we use the new fractional complex transform and the sub-equation method to study the nonlinear fractional differential equations and find the exact solutions. These solitary wave solutions demonstrate the fact that solutions to the perturbed nonlinear Schrodinger equation with power law nonlinearity model can exhibit a variety of behaviors.

Keywords: Time-fractional Burgers equation, Fractional calculus, sub-equation method.

1 INTRODUCTION

With the availability of symbolic computation packages like Maple or Mathematica, direct searching for exact solutions of nonlinear systems of partial differential equations (PDEs) has become more and more attractive. Having exact solutions of nonlinear systems of PDEs makes it possible to study nonlinear physical phenomena thoroughly and facilitates testing the numerical solvers as well as aiding the stability analysis of solutions. Wide classes of analytical methods have been proposed for solving the fractional differential equations, such as the fractional sub-equation method [1–3], the first integral method [4], and the (G'/G)-expansion method [5, 6], which can be used to construct the exact solutions for some time and space fractional differential equations. Based on these methods, a variety of fractional differential equations have been investigated and solved. In this present paper we applied the new extension of sub-equation method for finding new exact solitary wave solutions for time-fractional Burgers equation in the following form,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial q}{\partial x} - v \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < \alpha \leq 1, \ t > 0. \tag{1.1}$$

Recently, a new modification of Riemann-Liouville derivative is proposed by Jumarie [7]:

$$D_x^a f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\varepsilon)^{-\alpha} (f(\varepsilon) - f(0))\, d\varepsilon, \qquad 0 < \alpha < 1$$

Department of Mathematics, Faculty of Sciences, Gonbad Kavous University, Gonbad, Iran.
E-mails: a.neirameh@gonbad.ac.ir; a.neirameh@gmail.com

and gave some basic fractional calculus formulae, for example, formulae (4–12) and (4–13) in [7]:

$$D_x^\alpha (u(x) v(x)) = v(x) D_x^\alpha (u(x)) + u(x) D_x^\alpha (v(x)),$$
$$D_x^\alpha (f(u(x))) = f_u'(u) D_x^\alpha (u(x)) = D_x^\alpha f(u) \left(u_x'\right)^\alpha, \tag{1.2}$$

The last formula – has been applied to solve the exact solutions to some nonlinear fractional order differential equations. If this formula were true, then we could take the transformation

$$\xi = x - \frac{kt^\alpha}{\Gamma(1+\alpha)}$$

and reduce the partial derivative

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} \text{ to } U'(\xi).$$

Therefore the corresponding fractional differential equations become the ordinary differential equations which are easy to study. But we must point out that Jumarie's basic formulae and are not correct, and therefore the corresponding results on differential equations are not true [8]. Fractional derivative is as old as calculus. The most popular definitions are [9–12]:

(i) Riemann-Liouville definition: If n is a positive integer and $\alpha \in [n-1, n)$ the α^{th} derivative of f is given by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

(ii) Caputo Definition. For $\alpha \in [n-1, n)$ the α derivative of f is

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0 \left(D_a^\alpha(1)\right) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known product rule

$$D_a^\alpha(fg) = f D_a^\alpha(g) + g D_a^\alpha(f)$$

(iii) All fractional derivatives do not satisfy the known product rule

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{f D_a^\alpha(g) - g D_a^\alpha(f)}{g^2}$$

(iv) All fractional derivatives do not satisfy the known quotient rule:

$$D_a^{\alpha}(fog)(t) = f^{\alpha}(g(t))\, g^{\alpha}(t).$$

(v) All fractional derivatives do not satisfy the chain rule: $D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f$ in general.

(vi) Caputo definition assumes that the function f is differentiable. Authors introduced a new definition of fractional derivative as follows [16]:

For $\alpha \in [0,\ 1)$, and $f : [0,\ \infty) \to \mathrm{R}$ let

$$T_{\alpha}(f)(t) = \lim_{\xi \to 0} \frac{f\left(t + \xi t^{1-\alpha}\right) - f(t)}{\xi}.$$

For $t > 0, \alpha \in (0, 1)$. T_{α} is called the conformable fractional derivative of f of order α [17–18].

Definition 1.1. *Let $f^{\alpha}(t)$ stands for $T_{\alpha}(f)(t)$. Hence*

$$f^{\alpha}(t) = \lim_{\xi \to 0} \frac{f\left(t + \xi t^{1-\alpha}\right) - f(t)}{\xi}.$$

If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{\alpha}(t)$ exists, then by definition

$$f^{\alpha}(0) = \lim_{t \to 0^+} f^{\alpha}(t).$$

We should remark that $T_{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}$. Further, this definition coincides with the classical definitions of R-L and of Caputo on polynomials (up to a constant multiple). One can easily show that T_{α} satisfies all the properties in the theorem [15–16].

Theorem 1.1. *Let $\alpha \in [0,\ 1)$ and f, g be α-differentiable at a point t. Then:*

(i) $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, *for all* $a, b \in \mathrm{R}$;

(ii) $T_{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}$, *for all* $\mu \in \mathrm{R}$;

(iii) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$;

(iv) $T_{\alpha}\left(\dfrac{f}{g}\right) = \dfrac{fT_{\alpha}(g) - gT_{\alpha}(f)}{g^2}$.

If, in addition, f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha}\dfrac{df}{dt}$.

Theorem 1.2. *Let $f : [0,\ \infty) \to \mathrm{R}$ be a function such that f is differentiable and also differentiable. Let g be a function defined in the range of f and also differentiable; then, one has the following rule* [17]*:*

$$T_{\alpha}(fog)(t) = t^{1-\alpha} g'(t)\, f'(g(t)).$$

The above rule is referred to as Atangana beta-rule. We will present new derivative for some special functions *(i)* $T_\alpha\left(e^{cx}\right) = cx^{1-\alpha}e^{cx}, \quad c \in \mathrm{R}.$

(ii) $T_\alpha\left(\sin bx\right) = bx^{1-\alpha}\cos bx, \quad b \in \mathrm{R}.$

(iii) $T_\alpha\left(\cos bx\right) = -bx^{1-\alpha}\sin bx, \quad b \in \mathrm{R}.$

(iv) $T_\alpha\left(\frac{1}{\alpha}x^\alpha\right) = 1.$

However, it is worth noting the following fractional derivatives of certain functions:

(i) $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t}.$

(ii) $T_\alpha\left(\sin \frac{1}{\alpha}t\right) = \cos \frac{1}{\alpha}t.$

(iii) $T_\alpha\left(\cos \frac{1}{\alpha}t\right) = -\sin \frac{1}{\alpha}t.$

Definition 1.2. *(Fractional Integral). Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in f$. Then the α-fractional integral of f is defined by,*

$$I_a^\alpha(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$$

if the Riemann improper integral exists. It is interesting to observe that the α-fractional derivative [15 – 16].

Theorem 1.3. *(Inverse property). Let $a \geq 0$, and $\alpha \in (0, 1)$. Also, let f be a continuous function such that $I_a^\alpha f$ exists. Then*

$$T_\alpha\left(I_a^\alpha f\right)(t) = f(t), \quad for\ t \geq a.$$

In this paper, we obtain the exact solution of the fractional perturbed nonlinear Schrodinger equation with power law nonlinearity by means of the sub-equation method. The sub-equation method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of the most direct and effective algebraic methods for finding exact solutions of nonlinear fractional partial differential equations (FPDEs). The method is based on the homogeneous balance principle and the Jumarie's modified Riemann-Liouville derivative of fractional order.

2 METHOD APPLIED

Suppose that nonlinear fractional partial differential equations, say, in three independent variable x, y and t is given by

$$G\left(u, D_t^\alpha u, D_x^\alpha u, D_y^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha}, D_t^\alpha D_x^\alpha u, \ldots\right) = 0, \qquad 0 < \alpha \leq 1. \tag{2.1}$$

where $D_x^\alpha u$, $D_y^\alpha u$ and $D_t^\alpha u$ are comformable fractional derivatives of u, $u(x, y, t)$ is an unknown function, G is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. This method consists of the following steps:

Step 2.1 Using a wave transformation

$$u = u(\xi), \qquad \xi = k\frac{x^\alpha}{\alpha} + l\frac{y^\alpha}{\alpha} + c\frac{t^\alpha}{\alpha}, \tag{2.2}$$

Where k and c are real constants. This enables us to use the following changes:

$$D_t^\alpha(.) = c\frac{d}{d\xi}, \quad D_x^\alpha(.) = k\frac{d}{d\xi}, \quad D_y^\alpha(.) = l\frac{d}{d\xi}, \quad D_x^{2\alpha}(.) = k^2\frac{d^2}{d\xi^2}.$$

Under the transformation (2.2), Eq. (2.1) becomes an ordinary differential equation

$$N(u, u', u'', u''', \ldots) = 0, \tag{2.3}$$

where $u' = \frac{du}{d\xi}$.

Step 2.2 We assume that the solution of Eq. (2.3) is of the form

$$u(\xi) = \sum_{i=0}^{m} a_i\,(m + F(\xi))^i + \sum_{i=m+1}^{2m} a_i\,(m + F(\xi))^{m-i}\,, \tag{2.4}$$

where $a_i (i = 1, 2, \ldots, n)$ are real constants to be determined later. $F(\xi)$ expresses the solution of the auxiliary ordinary differential equation

$$F'(\xi) = \gamma(t)\,F^2(\xi) + \beta(t)\,F(\xi) + \alpha(t)\,, \tag{2.5}$$

Eq. (2.5) admits the following solutions:

$$\begin{aligned}
F(\xi) &= \begin{cases} -\sqrt{-b}\tanh(\sqrt{-b}\xi), & b < 0 \quad (9a) \\ -\sqrt{-b}\coth(\sqrt{-b}\xi), & b < 0 \quad (9b) \end{cases} \\
F(\xi) &= \begin{cases} \sqrt{b}\tan(\sqrt{b}\xi), & b > 0 \quad (9c) \\ -\sqrt{b}\cot(\sqrt{b}\xi), & b > 0 \quad (9d) \end{cases} \\
F(\xi) &= -\tfrac{1}{\xi+\xi_0}, \qquad \xi_0 = \text{const}, \quad b = 0 \quad (9e)
\end{aligned} \tag{2.6}$$

Integer m in (2.4) can be determined by considering homogeneous balance between the nonlinear terms and the highest derivatives of $u(\xi)$in Eq. (2.3) polynomial in $F(\xi)$, equating each coefficient of the polynomial to zero yields a set of algebraic equations for a_i, k, c.

Step 2.3 Solving the algebraic equations obtained in Step 3, and substituting the results into (2.3), then we obtain the exact traveling wave solutions for Eq. (2.1).

3 APPLICATION TO THE TIME-FRACTIONAL BURGERS EQUATAION

Using a wave transformation

$$u(x,t) = U(\xi), \xi = kx - \frac{ct^{\alpha}}{\alpha}, \tag{3.1}$$

by substituting Eq. (3.1), into Eq. (1.1) is reduced into an ODE

$$-cU' + k\varepsilon UU' - k^2 vU'' = 0,$$

by integrating once, we find

$$\xi_0 - cU + \frac{1}{2}k\varepsilon U^2 - k^2 vU' = 0. \tag{3.2}$$

Balancing U' with U^2in Eq. (3.2) give

$$m + 1 = 2m \Leftrightarrow m = 1.$$

We then assume that Eq. (3.2) has the following formal solution:

$$U(\xi) = a_0 + a_1 (h + F) + a_2 (h + F)^{-1}, \tag{3.3}$$

by considering the $F(\xi) + h = \Psi$ in Eq. (3.3) we have

$$U(\xi) = a_0 + a_1\Psi + a_2\Psi^{-1}, \tag{3.4}$$

and

$$\Psi' = \gamma\Psi^2 + (\beta - 2\gamma h)\Psi + \gamma h^2 - \beta h + \alpha. \tag{3.5}$$

Substituting Eqs. (3.4) − (3.5) into Eq. (3.2) and collecting all terms with the same order of ψ^j together, we convert the left-hand side of Eq. (3.2) into a polynomial in F^j. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for a_0, a_1, a_2 and h. By solving these algebraic equations we have

$$\begin{aligned}
a_1 &= \frac{2kv\gamma}{\varepsilon} \\
a_0 &= \frac{-c - k^2 v\beta + 2k^2 v\gamma h}{k\varepsilon} \\
a_2 &= -\frac{2kv(\alpha + \gamma h^2 - \beta h)}{\varepsilon} \\
h &= \frac{1}{2}\frac{\beta}{\gamma}, \\
c &= \sqrt{2\xi_0 k\varepsilon - 16k^4 v^2\gamma\alpha + 4k^4 v^2\beta^2}
\end{aligned}$$

So from (3.1) we have solitary wave solutions of Eq. (1.1) as follows.

If $b < 0$

$$u_1(x,t) = \frac{-c-k^2 v\beta+2k^2 v\gamma h}{k\varepsilon} + \frac{2kv\gamma}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \sqrt{-b}\tanh\left[\sqrt{-b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right) - \frac{2kv(\alpha+\gamma h^2-\beta h)}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \sqrt{-b}\tanh\left[\sqrt{-b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right)^{-1},$$

and

$$u_2(x,t) = \frac{-c-k^2 v\beta+2k^2 v\gamma h}{k\varepsilon} + \frac{2kv\gamma}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \sqrt{-b}\coth\left[\sqrt{-b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right) - \frac{2kv(\alpha+\gamma h^2-\beta h)}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \sqrt{-b}\coth\left[\sqrt{-b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right)^{-1}.$$

If $b > 0$

$$u_3(x,t) = \frac{-c-k^2 v\beta+2k^2 v\gamma h}{k\varepsilon} + \frac{2kv\gamma}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma}\sqrt{b}\tan\left[\sqrt{b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right) - \frac{2kv(\alpha+\gamma h^2-\beta h)}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} + \sqrt{b}\tan\left[\sqrt{b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right)^{-1},$$

and

$$u_4(x,t) = \frac{-c-k^2 v\beta+2k^2 v\gamma h}{k\varepsilon} + \frac{2kv\gamma}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \sqrt{b}\cot\left[\sqrt{b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right) - \frac{2kv(\alpha+\gamma h^2-\beta h)}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \sqrt{b}\cot\left[\sqrt{b}\left(kx - \frac{\sqrt{2\xi_0 k-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha}{\alpha}\right)\right]\right)^{-1}.$$

If $b = 0$ we have solution of Eq. (1.1) as follow

$$u_5(x,t) = \frac{-c-k^2 v\beta+2k^2 v\gamma h}{k\varepsilon} + \frac{2kv\gamma}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \frac{\alpha}{kx\alpha - \sqrt{2\xi_0 k\varepsilon-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha+\xi_0\alpha}\right) - \frac{2kv(\alpha+\gamma h^2-\beta h)}{\varepsilon}\left(\frac{1}{2}\frac{\beta}{\gamma} - \frac{\alpha}{kx\alpha - \sqrt{2\xi_0 k\varepsilon-16k^4 v^2\gamma\alpha+4k^4 v^2\beta^2}t^\alpha+\xi_0\alpha}\right)^{-1}.$$

4 CONCLUSION

Now, we briefly summarize the results in this paper. Firstly, the fractional complex transform is extremely simple but effective for solving nonlinear fractional differential equations. Secondly, the sub-equation method for nonlinear fractional differential equations with fractional complex transform has its own advantages: direct, succinct, and basic; and it can be used for many other nonlinear equations. Thirdly, to our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

5 ACKNOWLEDGEMENTS

I would like to express thanks to the editor and anonymous referees for their useful and valuable comments and suggestions.

RESUMO. Neste artigo usamos uma nova transformação fracional complexa e o método da sub-equação para estudar equações diferenciais não lineares e encontrar soluções exatas. As soluções de onda encontradas mostram que as soluções da equação não linear de Schrodinger perturbada com um modelo não linear de lei das potências pode apresentar diversos comportamentos diferentes.

Palavras-chave: Equação de Burgers, cálculo fracionário, método sub-equação.

REFERENCES

[1] S. Zhang & H.Q. Zhang. *Phys. Lett. A*, **375** (2011), 1069.

[2] B. Tong, Y. He, L. Wei & X. Zhang. *Phys. Lett. A*, **376** (2012), 2588.

[3] S. Guo, L. Mei, Y. Li & Y. Sun. *Phys. Lett. A*, **376** (2012), 407.

[4] B. Lu. *J. Math. Anal. Appl.*, **395** (2012), 684–693.

[5] B. Zheng. *Commun. Theor. Phys.*, **58** (2012), 623.

[6] K.A. Gepreel & S. Omran. *Chin. Phys. B*, **21** (2012), 110–204.

[7] G. Jumarie. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. *Computers and Mathematics with Applications*, **51** (2006), 1367–1376.

[8] C-s. Liu. Counter examples on Jumarie's two basic fractional calculus formulae. *Communications in Nonlinear Science and Numerical Simulation*, **22**(3) (2015), 9294.

[9] I. Podlubny. Fractional Diferential Equations, Academic Press, (1999).

[10] S.G. Samko, A.A. Kilbas & O.I. Marichev. Fractional Integrals and Derivatives: Theory and Applications, *Gordon and Breach Science Publishers*, (1993).

[11] A.A. Kilbas, M.H. Srivastava & J.J. Trujillo. Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, (2006).

[12] A.A. Kilbas & M. Saigo. On solution of integral equation of Abel-Volterra type. *Diff. Integr. Equat.*, **8**(5) (1995), 993–1011.

[13] T. Abdeljawad. On conformable fractional calculus. *Journal of Computational and Applied Mathematics*, **279** (2015), 57–66.

[14] T. Abdeljawad, M. Al Horani & R. Khalil. Conformable fractional semigroup operators. *Journal of Semigroup Theory and Applications*, (2015), Article 7, 1–9.

[15] M. Abu Hammad & R. Khalil. Conformable heat differential equation. *International Journal of Pure and Applied Mathematics*, **94**(2) (2014), 215–221.

[16] R. Khalil, M. Al Horani, A. Yousef & M. Sababheh. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, **264** (2014), 65–70.