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On a Regularising Convex Potential Related to a Variational Formulation of an Elastoplastic-Damage Constitutive Model

A.R. BALBO[1], Departamento de Matemática, UNESP, Av. Eng. Luiz Edmundo C. Coube, s/n, 17033-360 Bauru - SP, Brasil.

S.P.B. PROENÇA[2], Departamento de Estruturas, Av. Trabalhador Sãocarlense, 400, , ICMSC-USP, 13566-590 São Carlos - SP, Brasil.

Abstract. In this work we have proposed the definition of a regularising convex potential to be used in numerical analysis involving a certain class of constitutive elastoplastic-damage models. All the mathematical aspects discussed here are based on convex analysis, aiming at a variational formulation of the regularising elastopastic-damage potential and its conjugate potential. It is shown that the constitutive relations for the considered class of damage models are derived from those potentials by means of the respective sub-differentials sets. Furthermore, the potentials are defined in such a way that the complementarity and consistency conditions present in the local form of the damage model are satisfied. The optimality conditions of the resulting minimisation problem represents, in particular, a linear complementary problem. The numerical integration of the latter set of equations is exact if the time step does not includes damage followed by unloading.

1. Introduction

If one considers an ideal elastoplastic-damage material, in agreement with the results found in [1] and [9], whose constitutive response is composed by an initial linear elastoplastic regime followed by a linear softening domain, the associated potential is non-convex, since it is formed by the addition of a convex part related to the elastoplastic response plus a concave part corresponding to the linear softening regime. Such a characteristic implies difficulties because most of the numerical strategies aiming to verify the constitutive models are based on the minimization of the potential. In order to recover numerical efficiency of the optimization algorithms and following [5], this work proposes a regularising convex potential, which allows to verify the elastoplastic-damage constitutive relations in a step by step procedure.

[1]Acknowledgement: the work of the first author was partially supported by Fundunesp (Process 353/2001). E-mail: arbalbo@fc.unesp.br
[2]E-mail: persival@sc.usp.br

The variational formulation of the regularising potential and its conjugate is based on Convex Analysis Concepts, in [4] and [10]. The use of this name is justified by the fact that the potential is defined by the difference between a strictly convex elastoplastic potential and a damage potential which is convex on the damage variable. The optimality conditions for the potential minimization problem verifies the conditions of the linear softening constitutive model. In section **2** the local form of the constitutive relations for an ideal material is presented. Section **3** deals with the rate variational formulation of the same model. In the section **4** the regularising convex damage potential is presented and its properties are discussed. Section **5** shows the existence of the conjugate (dual) potential. In section **6** a more convenient form of the potential is written in terms of finite increments of strains and damage multiplier aiming the numerical treatment. Finally, in section **7** an extension of the potential to include the non-associative case is suggested.

2. Constitutive Relations in Rates to Elastoplastic-Damage Material

In what follows, it is assumed that the continuous body occupies a region B in the Euclidean pointwise space (Hilbert Space with finite dimension), being Γ_u and Γ_s its complementary boundaries where displacements and loads are prescribed respectively. A regime of small strains is considered and the ideal constitutive behaviour of the material presents a linear elastoplastic followed by a linear softening domain. Locally, the damage induces reduction of rigidity and permanent strains do not occurring. As a general upper bounded a quantity $\tau \geq 0$ of energy is assumed to be dissipated in correspondence to the damage processes and the, in accordance to [6], elastoplastic-damage rigidity E is a function of the fracture work or dissipated energy τ. The proposed model is related to the pure elastoplastic-damage case. In order to consider an evolution process, at any point $x \in \mathbf{B}$ the local form of the constitutive relations may be expressed in rates, based in [2], [3], [8] and, mainly, in [1] and [9], as follows:

$$\dot{\sigma} = \mathbf{E}(\tau)\dot{\epsilon} + \dot{\mathbf{E}}(\tau)\epsilon = \dot{\sigma}^{\mathbf{e}} + \dot{\sigma}^{\mathbf{d}}; \tag{2.1}$$

$$\mathbf{f}(\epsilon, \tau) \leq 0; \tag{2.2}$$

$$\tau - \bar{\tau} \leq 0 \Longleftrightarrow \mathbf{g}(\alpha, \tau) = -\alpha - (\tau - \bar{\tau}) \leq 0 \ , \ \alpha \geq 0; \tag{2.3}$$

$$\dot{\sigma}^{\mathbf{d}} = \dot{\lambda}\mathbf{f}_{\epsilon}(\epsilon, \tau); \tag{2.4}$$

$$\dot{\tau} = \dot{\lambda}\mathbf{r}(\epsilon, \tau); \tag{2.5}$$

$$\mathbf{f} \leq 0, \ \dot{\lambda} \geq 0, \ \mathbf{f}\dot{\lambda} = 0; \tag{2.6}$$

$$\text{if } \mathbf{f} = 0, \ \text{ then } \dot{\mathbf{f}}\dot{\lambda} = 0, \ \dot{\mathbf{f}} \leq 0 \ . \tag{2.7}$$

In relations above $\dot{\sigma}^d$ is the relaxed stress rate tensor due to damage effects, the scalar function $\mathbf{f}$ is a criteria for elastoplastic-damage evolution and represents an upper bound limit for the dissipated energy. The tensor $\mathbf{f}_{\epsilon}(\epsilon, \tau) \geq 0$ is considered to be normal to the surface defined by the evolution criteria $\mathbf{f}$ and $\mathbf{r}(\epsilon, \tau) \geq 0$ is a scalar function which contains a record of the previous irreversible history. The tensor $\mathbf{f}_\epsilon$ may be assumed in the non-associative case as $\mathbf{h}(\epsilon, \tau)$, considered to be normal to the surface of an elastoplastic-damage potential. The complementary and consistency conditions (2.6) and (2.7) account for the irreversibility of the process, respectively. Considering the relation (2.3), a scalar slack variable $\alpha \geq 0$ is introduced meaning the quantity of energy which remains to be dissipated. Thus, an additional complementary condition may be stated:

$$\mathbf{g}\alpha = 0 \text{ with } \mathbf{g} \leq 0 \text{ and } \alpha \geq 0; \tag{2.8}$$

$$\dot{\mathbf{g}} = -\dot{\alpha} - \dot{\tau} \text{ and } \dot{\mathbf{g}} \leq 0 \text{ then } \dot{\tau} \leq \dot{\alpha} \leq 0. \tag{2.9}$$

In particular, if $\dot{\mathbf{g}} = 0$ then $\dot{\alpha} = -\dot{\tau}$.

The "damage multiplier" follows from the consistency condition (2.7), by considering the relations (2.2) and (2.5):

$$\dot{\mathbf{f}} = \mathbf{f}_\epsilon \cdot \dot{\epsilon} + \mathbf{f}_\tau \dot{\tau} = \mathbf{f}_\epsilon \cdot \dot{\epsilon} - \dot{\lambda}\mathbf{f}_\tau \mathbf{r}(\epsilon, \tau) = 0. \tag{2.10}$$

Thus,

$$\dot{\lambda} = \frac{(\mathbf{f}_\epsilon \cdot \dot{\epsilon})}{\mathbf{G}}; \tag{2.11}$$

where $\mathbf{G} = \mathbf{f}_\tau r(\epsilon, \tau) \geq 0$ is the elastoplastic-damage modulus, assumed to be positive. Furthermore, $\mathbf{f}_\epsilon \cdot \dot{\epsilon} \geq 0$ indicates that damage evolution occurs when the deformation rate appoints to the outside of the elastoplastic domain. By substitution of (2.11) in (2.4), the relation for $\dot{\sigma}^d$ becomes:

$$\dot{\sigma}^d = -\frac{(\mathbf{f}_\epsilon \otimes \mathbf{f}_\epsilon)}{\mathbf{G}}\dot{\epsilon}. \tag{2.12}$$

In the general case $\dot{\sigma}^d$ is non-symmetric. This fact can be recovered by substitution of $\mathbf{h}$ by $\mathbf{f}$ in the relation (2.12). By combining (2.1) and (2.12), the constitutive relation can be expressed for the associative case as :

$$[\mathbf{E}(\tau) - \frac{(\mathbf{f}_\epsilon \otimes \mathbf{f}_\epsilon)}{\mathbf{G}}]\dot{\epsilon} \text{ if } \dot{\lambda} \geq 0. \tag{2.13}$$

Using the relations (2.1) and (2.11), the fracture work evolution can be expressed as:

$$\dot{\tau} = [\frac{(\mathbf{f}_\epsilon \cdot \dot{\epsilon})}{\mathbf{G}}]\mathbf{r}(\epsilon, \tau). \tag{2.14}$$

Considering (2.8), in particular, $\dot{\mathbf{g}} = 0$ if $\dot{\alpha} = -\dot{\tau}$.

Furthermore, it is possible to find a relation between $\dot{\alpha}$ and $-\dot{\tau}$, expressed by

$$\dot{\lambda} = \psi\dot{\alpha} \text{ with } \psi = \mathbf{r}^{-\mathbf{1}}, \tag{2.15}$$

where $\psi = \frac{(\mathbf{f}_\epsilon \otimes \dot{\epsilon})\mathbf{E}_\tau}{\|\mathbf{f}_\epsilon\|^\mathbf{2}}$ for the associative case.

3. Rate Variational Formulation of the Constitutive Model.

In what follows some hypothesis, based in [5], are assumed :

i) Let $\mathbf{W}^*$ and $\mathbf{W}$ not empty dual vectorial subspaces in $\mathbf{B}$, containing respectively the rate of stresses and strains tensors. The space $\mathbf{B}$ and the subspaces $\mathbf{W}^*$ and $\mathbf{W}$ are endowed with norms:

$$\|\mathbf{x}\|_{\mathbf{B}} = (\int_{\mathbf{B}} \mathbf{x}\cdot\mathbf{x}\ \mathbf{dB})^{\frac{1}{2}};\ \|\dot{\epsilon}\|_{\mathbf{W}} = (\int_{\mathbf{B}} \dot{\epsilon}\cdot\dot{\epsilon}\ \mathbf{dB})^{\frac{1}{2}};\ \|\dot{\sigma}\|_{\mathbf{W}^*} = (\int_{\mathbf{B}} \dot{\sigma}\cdot\dot{\sigma}\ \mathbf{dB})^{\frac{1}{2}}. \quad (3.1)$$

Between such spaces a duality product is introduced and defined as:

$$\langle\dot{\sigma},\dot{\epsilon}\rangle = \langle\dot{\sigma},\dot{\epsilon}\rangle_{\mathbf{W}^*\times\mathbf{W}} = \textstyle\int_{\mathbf{B}} \dot{\sigma}\cdot\dot{\epsilon}\ \mathbf{dB}, \quad (3.2)$$

with $\mathbf{x}\in\mathbf{B}$, $\dot{\epsilon}\in\mathbf{W}$ and $\dot{\sigma}\in\mathbf{W}^*$.

ii) In the consistency and complementary relations (2.6) and (2.7), $\mathbf{f}$ and $\dot{\lambda}$ are imposed to be scalars. Then, the damage multiplier $\dot{\lambda}$ and the damage criteria $\mathbf{f}$ are defined in the dual spaces $\mathbf{\Lambda}$, $\mathbf{\Lambda}^*$ in $\mathbf{B}$, respectively, with norms expressed by:

$$\|\dot{\lambda}\|_{\mathbf{\Lambda}} = \textstyle\int_{\mathbf{B}} |\dot{\lambda}|\ \mathbf{dB};\ \|\ \mathbf{f}\|_{\mathbf{\Lambda}^*} = \int_{\mathbf{B}} |\mathbf{f}|\ \mathbf{dB}. \quad (3.3)$$

Between the spaces $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^*$, a duality product is introduced and defined as:

$$\langle\mathbf{f},\dot{\lambda}\rangle = \langle\mathbf{f},\dot{\lambda}\rangle_{\mathbf{\Lambda}^*\times\mathbf{\Lambda}} = \textstyle\int_{\mathbf{B}} \mathbf{f}\dot{\lambda}\ \mathbf{dB}; \quad (3.4)$$

The following sets are convenient to define:

$$\mathbf{\Lambda}^+_{\mathbf{f}} = \{\dot{\lambda}\geq 0/\ \mathbf{f}\dot{\lambda}=0,\ \forall\mathbf{x}\in\mathbf{B}\}; \quad (3.5)$$

$$\mathbf{\Lambda}_{\mathbf{f}} = \{\dot{\lambda}\geq 0,\ \forall\mathbf{x}\in\mathbf{B}\}; \quad (3.6)$$

$$\mathbf{\Lambda}_{\mathbf{g}} = \{\dot{\alpha}\geq 0,\ \forall\mathbf{x}\in\mathbf{B}\}. \quad (3.7)$$

The definitions (3.3) until (3.6) could be stablished to include the general case where $\mathbf{f}$ and $\dot{\lambda}$ would be vectors.

iii) $\mathbf{f}=\mathbf{f}(\epsilon,\tau)$ is a regular (non-strictly) convex scalar function of the field $\epsilon\in\mathbf{W}$ and the scalar $\tau\in\Re^+$; $\mathbf{f}_\epsilon=\mathbf{f}_\epsilon(\epsilon,\tau)$ is a linear operator of $\mathbf{W}\times\Re^+$ in $\mathbf{W}^*\times\Re^+$, assumed as: iii.1) lower and upper bounded in $\mathbf{\Lambda}_{\mathbf{f}}$, i.e., there are constants $h_0>0$ and $h_1>0$ such that

$$h_0\|\dot{\lambda}\|_{\mathbf{\Lambda}} \geq \|\dot{\lambda}\mathbf{f}_\epsilon\|_{\mathbf{W}^*} \geq h_1\|\dot{\lambda}\|_{\mathbf{\Lambda}},\ \forall\lambda\in\mathbf{\Lambda}_{\mathbf{f}}. \quad (3.8)$$

This property implies $\|\dot{\sigma}^{\mathbf{d}}\| = \|\dot{\lambda}\mathbf{f}_\epsilon\|_{\mathbf{W}^*}\neq 0$ and finite. Moreover, the upper bound also implies that there is only one $\lambda\in\mathbf{\Lambda}_{\mathbf{f}}$ such that $\mathbf{f}_\epsilon\dot{\lambda}$ is equal to a prescribed $\dot{\sigma}^d$.

iii.2) $\dot{\lambda}\mathbf{f}_\epsilon$ is continuously dependent on $\dot{\epsilon}\in\mathbf{W}$, i.e., for $h_2>0$,

$$|\ \langle\dot{\epsilon},\dot{\lambda}\mathbf{f}_\epsilon\rangle\ | \leq h_2\|\dot{\epsilon}\|_{\mathbf{W}}\|\dot{\lambda}\|_{\mathbf{\Lambda}};\ \forall\dot{\lambda}\in\mathbf{\Lambda},\ \forall\dot{\epsilon}\in\mathbf{W}. \quad (3.9)$$

The property (3.9) ensures that the rate of dissipated energy $| \langle \dot{\epsilon}, \dot{\lambda} \mathbf{f}_\epsilon \rangle |$ is finite. Then the damage is suposed to be a continuous and bounded process.

iv) The elastoplastic modulus tensor $\mathbf{E}$ is symmetric and positive definite. Then, for $h_3 \geq 0$ and $h_4 \geq \|\mathbf{E}\|_\infty$, the following condition is valid:

$$h_3 \|\dot{\epsilon}\|^2_{\mathbf{W}} \leq \langle \mathbf{E}\dot{\epsilon}, \dot{\epsilon} \rangle \leq h_4 \|\dot{\epsilon}\|^2_{\mathbf{W}}; \; \forall \dot{\epsilon} \in \mathbf{W}. \tag{3.10}$$

v) The elastoplastic-damage modulus $\mathbf{G}$, defined in (2.11), is a non-negative number, in such a way, for $h_5 \geq 0$ and $h_6 \geq \mathbf{G}$,

$$h_5 \|\dot{\lambda}\|^2_{\mathbf{\Lambda}} \leq \langle \mathbf{G}\dot{\lambda}, \dot{\lambda} \rangle \leq h_6 \|\dot{\lambda}\|^2_{\mathbf{\Lambda}}; \; \forall \dot{\lambda} \in \mathbf{\Lambda_f}. \tag{3.11}$$

For the general elastoplastic-damage model, $\mathbf{G}$ would be a semi-positive definite operator.

vi) For $\lambda \in \mathbf{\Lambda}^+_{\mathbf{f}}$, the complementary and consistency conditions (2.6) and (2.7), respectively, may be expressed in an equivalent way as:

$$\langle \dot{\mathbf{f}}, \dot{\lambda^*} - \dot{\lambda} \rangle = \langle \mathbf{f}_\epsilon \cdot \dot{\epsilon} - \dot{\lambda} \mathbf{f}_\tau \mathbf{r}(\epsilon, \tau), \dot{\lambda^*} - \dot{\lambda} \rangle \leq 0; \; \forall \dot{\lambda^*} \in \mathbf{\Lambda}^+_{\mathbf{f}}. \tag{3.12}$$

4. A Regularising Convex Potential for the Elastoplastic-Damage Model.

Initially, let a potential $\mathbf{\Phi} : \mathbf{W} \times \mathbf{\Lambda}^+ \longrightarrow \Re$ be defined as :

$$\mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda}) = \tfrac{1}{2} \langle \mathbf{E}\dot{\epsilon}, \dot{\epsilon} \rangle + \langle -\dot{\lambda} \mathbf{f}_\epsilon, \dot{\epsilon} \rangle + \tfrac{1}{2} \langle \mathbf{G}\dot{\lambda}, \dot{\lambda} \rangle. \tag{4.1}$$

For this potential, the following properties are valid :

Property 4.1 $\mathbf{\Phi}$ is convex in the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda}^+_{\mathbf{f}}$.

Justification: this fact is ensured directly from (3.10) and (3.11), as the operator $\mathbf{E}$ is symmetric, positive definite and the damage modulus $\mathbf{G}$ is non-negative.

Property 4.2 $\mathbf{\Phi}$ is continuum in the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda}^+_{\mathbf{f}}$.
Justification: due to (3.8), (3.9), (3.10) and (3.11), $\mathbf{\Phi}$ is continuously dependent on the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda}^+_{\mathbf{f}}$, then, $\mathbf{\Phi}$ is continuum on the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda}^+_{\mathbf{f}}$.

Property 4.3 $\mathbf{\Phi}$ is coercive in the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda}^+_{\mathbf{f}}$.

Justification: using the properties found in (3.9), (3.10) and (3.11), it is shown that:

$$\lim_{\|\dot{\epsilon}\| \longrightarrow +\infty} \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda}) = +\infty \text{ and } \lim_{|\dot{\lambda}| \longrightarrow +\infty} \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda}) = +\infty. \tag{4.2}$$

Hence, $\mathbf{\Phi}$ is coercive on the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda}^+_{\mathbf{f}}$.

Proposition 4.1 *The potential* $\mathbf{\Phi}$ *defined in* (4.1) *reaches its infimum in* $\mathbf{W}$ *and* $\mathbf{\Lambda_f^+}$ *, i.e., there is a solution for the following infimum problem,*

$$\inf_{\dot{\lambda}\in\mathbf{\Lambda_f^+}}\inf_{\dot{\epsilon}\in\mathbf{W}}\mathbf{\Phi}(\dot{\epsilon},\dot{\lambda})=\inf_{\dot{\epsilon}\in\mathbf{W}}\inf_{\dot{\lambda}\in\mathbf{\Lambda_f^+}}\mathbf{\Phi}(\dot{\epsilon},\dot{\lambda}). \tag{4.3}$$

Justification: Due to the convexity, continuity and coerciveness of $\mathbf{\Phi}$, justified in **Properties 4.1**, **4.2** and **4.3**, in the variables $\dot{\epsilon}\in\mathbf{W}$ and $\dot{\lambda}\in\mathbf{\Lambda_f^+}$, with $\mathbf{W}$ and $\mathbf{\Lambda_f^+}$ non-empties closed sets, then, according to results found in [4] and [10], $\mathbf{\Phi}$ is weakly lower semicontinuous and presents the growth property. As a consequence, $\mathbf{\Phi}$ is bounded in the variables $\dot{\epsilon}\in\mathbf{W}$ and $\dot{\lambda}\in\mathbf{\Lambda_f^+}$. Thus there is infimum to $\mathbf{\Phi}$ in the sets $\mathbf{W}$ and $\mathbf{\Lambda_f^+}$, thus, $\mathbf{\Phi}$ reaches its infimum in those sets, i. e., there is solution for the infimum problem (4.3).

Observation: The infimum form that appears by side right in the expression (4.3), where the $\dot{\epsilon}$ follows from a given $\dot{\lambda}$, is not usual to analyse the engineering structural problem.

Proposition 4.2 *If* $\mathbf{\Phi}$ *admit infimum in* $\mathbf{W}$ and $\mathbf{\Lambda_f^+}$ *, then* $\mathbf{\Phi}$ *is lower semicontinuous (l.s.c) in* $\mathbf{W}$ *and* $\mathbf{\Lambda_f^+}$.

Justification: Due to **Proposition 4.1**, there are $\dot{\epsilon}^*\in\mathbf{W}$ and $\dot{\lambda}^*\in\mathbf{\Lambda_f^+}$ be, assumed as infimum for $\mathbf{\Phi}$. Then, $\mathbf{\Phi}$ is l.s.c. in $\dot{\epsilon}^*\in\mathbf{W}$ and in $\dot{\lambda}^*\in\mathbf{\Lambda_f^+}$ so, to $n\longrightarrow+\infty$:

$$\lim_{\dot{\epsilon}^{\mathbf{n}}\longrightarrow\dot{\epsilon}^*}[\inf_{\dot{\epsilon}^{\mathbf{n}}\in\mathbf{W}}\inf_{\dot{\lambda}\in\mathbf{\Lambda_f^+}}\mathbf{\Phi}(\dot{\epsilon}^{\mathbf{n}},\dot{\lambda})]\geq\mathbf{\Phi}(\dot{\epsilon}^*,\dot{\lambda}^*)$$

and

$$\lim_{\dot{\lambda}^{\mathbf{n}}\longrightarrow\dot{\lambda}^*}[\inf_{\dot{\lambda}^{\mathbf{n}}\in\mathbf{\Lambda_f^+}}\inf_{\dot{\epsilon}\in\mathbf{W}}\mathbf{\Phi}(\dot{\epsilon},\dot{\lambda}^{\mathbf{n}})]\geq\mathbf{\Phi}(\dot{\epsilon}^*,\dot{\lambda}^*).$$

Thus, $\mathbf{\Phi}$ is l.s.c in $\mathbf{W}$ and $\mathbf{\Lambda_f^+}$.

Proposition 4.3 *The potential* $\mathbf{\Phi}$ *is proper in the variables* $\dot{\epsilon}\in\mathbf{W}$ *and* $\dot{\lambda}\in\mathbf{\Lambda_f^+}$.

Justification: $\mathbf{\Phi}$ is proper in the variable $\dot{\epsilon}\in\mathbf{W}$ and in $\dot{\lambda}\in\mathbf{\Lambda_f^+}$ so, assuming $\dot{\epsilon}^*\in\mathbf{W}$ and $\dot{\lambda}^*\in\mathbf{\Lambda_f^+}$ as infimums, respectively, in $\mathbf{W}$ and $\mathbf{\Lambda_f^+}$, due to (3.9) and the **Proposition 4.1**:

$$\mathbf{\Phi}(\dot{\epsilon},\dot{\lambda})\geq\mathbf{\Phi}(\dot{\epsilon}^*,\dot{\lambda}*)=\frac{1}{2}\langle\mathbf{E}\dot{\epsilon},\dot{\epsilon}\rangle+\langle-\dot{\lambda}\mathbf{f}_\epsilon,\dot{\epsilon}\rangle+\frac{1}{2}\langle\mathbf{G}\dot{\lambda},\dot{\lambda}\rangle\geq-\infty.$$

Thus, $\mathbf{\Phi}$ is proper in the variable $\dot{\epsilon}\in\mathbf{W}$ and in $\dot{\lambda}\in\mathbf{\Lambda_f^+}$.

Proposition 4.4 *The potential* $\mathbf{\Phi}$ *is differentiable in the variables* $\dot{\epsilon}\in\mathbf{W}$ *and* $\dot{\lambda}\in\mathbf{\Lambda_f^+}$.

Justification: by using the total differential absolute value with relation to the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda_f^+}$ and considering $\dot{\epsilon^*} \in \mathbf{W}$ and $\dot{\lambda^*} \in \mathbf{\Lambda_f^+}$ such that $\dot{\epsilon^*} \longrightarrow \dot{\epsilon}$ and $\dot{\lambda^*} \longrightarrow \dot{\lambda}$, then, due to (3.9), (3.10) and (3.11):

$$|\mathbf{\Phi}(\dot{\epsilon^*}, \dot{\lambda^*}) - \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda}) - \langle \mathbf{E}\dot{\epsilon}, \dot{\epsilon^*} - \dot{\epsilon} \rangle - \langle -\dot{\lambda}\mathbf{f}_\epsilon, \dot{\epsilon^*} - \dot{\epsilon} \rangle - \langle -(\dot{\lambda^*} - \dot{\lambda})\mathbf{f}_\epsilon, \dot{\epsilon} \rangle - \langle \mathbf{G}\dot{\lambda}, \dot{\lambda^*} - \dot{\lambda} \rangle|$$

$$\leq \frac{1}{2} h_3 \|\dot{\epsilon} * - \dot{\epsilon}\|^2_{\mathbf{B}} + h_2 \|\dot{\epsilon^*} - \dot{\epsilon}\|_{\mathbf{B}} \|\dot{\lambda^*} - \dot{\lambda}\|_{\mathbf{B}} + \frac{1}{2} h_6 \|\dot{\lambda^*} - \dot{\lambda}\|^2_{\mathbf{B}}.$$

The latter condition implies that $\mathbf{\Phi}$ is differentiable for $\dot{\epsilon^*} \longrightarrow \dot{\epsilon}$ and $\dot{\lambda^*} \longrightarrow \dot{\lambda}$. Thus, $\mathbf{\Phi}$ is differentiable in the variables $\dot{\epsilon} \in \mathbf{W}$ and $\dot{\lambda} \in \mathbf{\Lambda_f^+}$.

Proposition 4.5 *If the potential $\mathbf{\Phi}$ is convex, l.s.c., proper and differentiable in the variable $\dot{\lambda} \in \mathbf{\Lambda_f^+}$, then the consistency conditions of the elastoplastic-damage model, included in* (3.12) *are satisfied.*

Justification: Since that $\mathbf{\Phi}$ is convex, l.s.c., proper and differentiable in the variable $\dot{\lambda^*} \in \mathbf{\Lambda_f^+}$, then there is $\dot{\lambda^*} \in \mathbf{\Lambda_f^+}$, assumed as infimum to $\mathbf{\Phi}$ in $\mathbf{\Lambda_f^+}$, satisfying the optimality conditions:

$$\begin{aligned} &\langle \nabla \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda^*}), \dot{\lambda^*} - \dot{\lambda} \rangle \geq 0; \, \forall \dot{\lambda^*} \in \mathbf{\Lambda_f^+} \\ \Longleftrightarrow &\langle \mathbf{f}_\epsilon \cdot \dot{\epsilon} - \dot{\lambda^*} \mathbf{f}_\tau \mathbf{r}(\epsilon, \tau), \dot{\lambda^*} - \dot{\lambda} \rangle \leq 0; \, \forall \dot{\lambda^*} \in \mathbf{\Lambda_f^+}; \end{aligned} \tag{4.4}$$

which is equivalent to consistency conditions (3.12) for the elastoplastic-damage model.

5. Existence of the Conjugate Potential $\mathbf{\Phi}^*$

Since $\mathbf{\Phi}$, written in the variable $\dot{\epsilon} \in \mathbf{W}$, for some $\dot{\lambda} \in \mathbf{\Lambda_f^+}$, is a convex potential, l.s.c., proper, defined in a non-empty set $\mathbf{W}$, there is a conjugated potential $\mathbf{\Phi}^* : \mathbf{W}^* \times \mathbf{\Lambda_f^+} \longrightarrow \Re$, verifying the following condition:

$$\mathbf{\Phi}^*(\dot{\sigma}, \dot{\lambda^*}) = \sup_{\dot{\epsilon} \in \mathbf{W}} \{ \langle \dot{\sigma}, \dot{\epsilon} \rangle - \Phi(\dot{\epsilon}, \dot{\lambda^*}) \}; \, \forall \dot{\sigma} \in \mathbf{W}^*. \tag{5.1}$$

This is in agreement with the results found in [4] and [10] , in the hypothesis of $\mathbf{W}^*$ to be non-empty set.

For $\mathbf{\Phi}^*(\dot{\sigma}, \dot{\lambda^*}) < \infty$, there are sub-differential sets $\partial_\epsilon \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda^*})$ and $\partial_\sigma \mathbf{\Phi}^*(\dot{\sigma}, \dot{\lambda^*})$, which are closed and non-empty, such that the following relations of duality may be established:

$$\dot{\sigma} \in \partial_\epsilon \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda^*}) \Longleftrightarrow \dot{\epsilon} \in \partial_\sigma \mathbf{\Phi}^*(\dot{\sigma}, \dot{\lambda^*}), \tag{5.2}$$

where, to $\dot{\lambda^*} \in \mathbf{\Lambda_f^+}$:

$$\partial_\epsilon \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda^*}) = \{ \gamma^* \in \mathbf{W}^* \colon \, \mathbf{\Phi}(\dot{\epsilon^*}, \dot{\lambda^*}) - \mathbf{\Phi}(\dot{\epsilon}, \dot{\lambda^*}) \geq \langle \dot{\epsilon^*} - \dot{\epsilon}, \gamma^* \rangle, \, \forall \dot{\epsilon^*} \in \mathbf{W} \}. \tag{5.3}$$

Analogously, it is defined the sub-differential $\partial_\sigma \mathbf{\Phi}^*(\dot\sigma, \dot{\lambda^*})$.
Since $\mathbf{\Phi}$ is differentiable in $\dot{\epsilon^*} \in \mathbf{W}$, then $\dot\sigma$ is uniquely determined by $\partial_\epsilon \mathbf{\Phi}(\dot\epsilon, \dot{\lambda^*})$.
Hence,

$$\dot\sigma = \nabla_\epsilon \mathbf{\Phi}(\dot\epsilon, \dot{\lambda^*}) = \mathbf{E}\dot\epsilon - \dot{\lambda^*}\mathbf{f}_\epsilon = \dot\sigma^e + \dot\sigma^d. \tag{5.4}$$

Thus, the following duality relation is valid:

$$\dot\sigma = \nabla_\epsilon \mathbf{\Phi}(\dot\epsilon, \dot{\lambda^*}) \Longleftrightarrow \dot\epsilon \in \partial_\sigma \mathbf{\Phi}^*(\dot\sigma, \dot{\lambda^*}). \tag{5.5}$$

The relation $\dot\epsilon = \nabla_\sigma \mathbf{\Phi}^*(\dot\sigma, \dot{\lambda^*})$ is not verified $\forall \dot\lambda \in \mathbf{\Lambda_f^+}$. In fact, it may exist $\dot\epsilon^{\mathbf{1}}$ and $\dot\epsilon^{\mathbf{2}}$ belonging to $\partial_\sigma \mathbf{\Phi}^*(\dot\sigma, \dot{\lambda^*})$ and correlated to one unique $\dot\sigma \in \mathbf{W}^*$.

Hence, $\mathbf{\Phi}^*$ is not always differentiable in $\dot\sigma \in \mathbf{W}^*$. Making use of (5.5) and of the result found in [4] and [10], the following relation is true for the potentials $\mathbf{\Phi}^*$ and $\mathbf{\Phi}$:

$$\mathbf{\Phi}^*(\dot\sigma, \dot{\lambda^*}) + \mathbf{\Phi}(\dot\epsilon, \dot{\lambda^*}) = \langle \dot\sigma, \dot\epsilon \rangle \Leftrightarrow \dot\sigma \in \partial_\epsilon \mathbf{\Phi}(\dot\epsilon, \dot{\lambda^*}) \text{ and } \dot\epsilon \in \partial_\sigma \mathbf{\Phi}^*(\dot\sigma, \dot{\lambda^*}).$$

In what follows, the incremental variational form of the model is presented, leading to a numerical treatment to the elastoplastic-damage model.

6. Incremental Variational Form.

The restrictive condition $\dot\lambda \in \mathbf{\Lambda_f^+}$ can be relaxed if one considers an indicator function $\mathbf{I}_{\mathbf{\Lambda_f^+}}$ defined as :

$$\mathbf{I}_{\mathbf{\Lambda_f^+}} = \begin{cases} 0 & \text{if } \dot\lambda \in \mathbf{\Lambda_f^+} \\ +\infty & \text{if } \dot\lambda \in \mathbf{\Lambda_f} - \mathbf{\Lambda_f^+}. \end{cases} \tag{6.1}$$

The indicator may be introduced into the model by means of the following asymptotic approximation :

$$\mathbf{I}_{\mathbf{\Lambda_f^+}} = \langle (\frac{-1}{\delta})\mathbf{f}, \dot\lambda \rangle_{\mathbf{\Lambda^*\times\Lambda}} = \langle (\frac{-1}{\delta})\mathbf{f}, \dot\lambda \rangle = (\frac{-1}{\delta}) \int_{\mathbf{B}} \mathbf{f} \cdot \dot\lambda \mathbf{dB} \text{ with } \delta \longrightarrow 0^+. \tag{6.2}$$

Thus, by subtraction of $\mathbf{I}_{\mathbf{\Lambda_f^+}}$ defined in (6.1) in the equation (4.1) and considering the approximation (6.2), for $\dot\lambda \in \mathbf{\Lambda_f}$, the potential defined becomes:

$$\mathbf{\Phi}_\delta(\dot\epsilon, \dot\lambda) = \frac{1}{2}\langle \mathbf{E}\dot\epsilon, \dot\epsilon \rangle + \langle -\dot\lambda \mathbf{f}_\epsilon, \dot\epsilon \rangle + \frac{1}{2}\langle \mathbf{G}\dot\lambda, \dot\lambda \rangle - \langle (\frac{-1}{\delta})\mathbf{f}, \dot\lambda \rangle \tag{6.3}$$

for all $\dot\lambda \in \mathbf{\Lambda_f}$ and $\delta \longrightarrow 0^+$. As $\delta \longrightarrow 0^+$ then $\dot\sigma \in \mathbf{\Phi}_\delta(\dot\epsilon, \dot\lambda)$ converges to $\dot\sigma \in \mathbf{\Phi}(\dot\epsilon, \dot\lambda)$.

Finally, by substitution of $\dot\lambda = \psi\dot\alpha$ defined in (2.15), one arrives to the equivalent potential:

$$\mathbf{\Phi}_\delta(\dot\epsilon, \dot\alpha) = \frac{1}{2}\langle \mathbf{E}\dot\epsilon, \dot\epsilon \rangle - \langle \psi\dot\alpha \mathbf{f}_\epsilon, \dot\epsilon \rangle + \frac{1}{2}\langle \mathbf{G}\psi\dot\alpha, \psi\dot\alpha \rangle + \langle (\frac{1}{\delta})\mathbf{f}, \psi\dot\alpha \rangle \tag{6.4}$$

for all $\dot{\alpha} \in \mathbf{\Lambda_g}$, with $\delta \longrightarrow 0^+$, and noting that $\dot{\lambda} \in \mathbf{\Lambda_f}$ implies $\dot{\alpha} \in \mathbf{\Lambda_g}$. An incremental variational form results from a time discretization, which is expressed as

$$\triangle\sigma = \dot{\sigma}\triangle\mathbf{t}; \quad \triangle\epsilon = \dot{\epsilon}\triangle\mathbf{t}; \quad \triangle\alpha = \dot{\alpha}\triangle\mathbf{t}. \tag{6.5}$$

By substitution of (6.5) into (6.4), with $\triangle\alpha \in \mathbf{\Lambda_g}$, the following potential results as a function of incremental variables:

$$\begin{aligned} &\mathbf{\Phi}_\delta(\triangle\epsilon, \triangle\alpha) \\ &\quad = \tfrac{1}{2}\langle\mathbf{E}\triangle\epsilon,\triangle\epsilon\rangle - \langle\psi\triangle\alpha\mathbf{f}_\epsilon,\triangle\epsilon\rangle + \tfrac{1}{2}\langle\mathbf{G}\psi\triangle\alpha,\psi\triangle\alpha\rangle + \langle\triangle t(\tfrac{\mathbf{f}}{\delta}),\psi\triangle\alpha\rangle. \end{aligned} \tag{6.6}$$

In particular, taking $\delta = \triangle t$, an extended potential results:

$$\mathbf{\Phi_f}(\triangle\epsilon, \triangle\alpha) = \frac{1}{2}\langle\mathbf{E}\triangle\epsilon,\triangle\epsilon\rangle - \langle\psi\triangle\alpha\mathbf{f}_\epsilon,\triangle\epsilon\rangle + \frac{1}{2}\langle\mathbf{G}\psi\triangle\alpha,\psi\triangle\alpha\rangle + \langle\mathbf{f},\psi\triangle\alpha\rangle. \tag{6.7}$$

Locally , with $\triangle\lambda \in \mathbf{\Lambda_f}$ or $\triangle\alpha \in \mathbf{\Lambda_g}$, considering the potential $\mathbf{\Phi_f}$ in the relation (4.3), the optimality conditions, for that infimum problem, are equivalent to:

$$[\mathbf{f} + \mathbf{f}_\epsilon \cdot \triangle\epsilon - \mathbf{G}\triangle\lambda] = [\mathbf{f} + \mathbf{f}_\epsilon \cdot \triangle\epsilon - \mathbf{G}\psi\triangle\alpha] \leq 0; \tag{6.8}$$

$$[\mathbf{f} + \mathbf{f}_\epsilon \cdot \triangle\epsilon - \mathbf{G}\triangle\lambda] \cdot \triangle\lambda = [\mathbf{f} + \mathbf{f}_\epsilon \cdot \triangle\epsilon - \mathbf{G}\psi\triangle\alpha] \cdot \psi\triangle\alpha = 0. \tag{6.9}$$

This fact happens due to local relations (6.8) and (6.9) to be equivalent to the linear complementarity problem (3.11), which can be solved by mathematical programming methods. In particular, if $\mathbf{f}$ is piecewise linear, an algorithm able to solve (6.8) gives exact increments $\triangle\lambda$ or $\triangle\alpha$, which verify $\mathbf{f} = 0$ at a step $t + \triangle t$. As a consequence, the constitutive relation is represented in an exact way for any $\triangle t$ which does not implies damage followed by unloading.

7. Extension to the Non-Associative Case.

The potentials $\mathbf{\Phi}$, $\mathbf{\Phi}_\delta$ and $\mathbf{\Phi_f}$ defined in (4.3), (6.3) and (6.7), respectively, may be defined in order to include the non-associative case. This is done by substitution of the gradient function $\mathbf{f}_\epsilon$ by tensor $\mathbf{h}$, which is normal to a damage potential surface, such that $\dot{\sigma}^d$ is redefinite by $\dot{\sigma}^d = -\dot{\lambda}\mathbf{h} = \psi\dot{\alpha}\mathbf{h}$.

8. Conclusion

A regularising convex damage potential has been defined. Applying convexity concepts and assuming some properties, the existence of a convex conjugate potential was proved. Also, it was shown that from the respective sub-differential sets of that potential it is possible to derive the constitutive relations in rates, including the complementarity and consistency relations. The incremental form of the relations were then obtained aiming the numerical treatment of the elastoplastic-damage model and outlined a possibility for an exact numerical integration if linear softening is assumed and the displacement increment does not violate the limit for the total dissipated energy. The local relations are equivalent to a linear complementarity problem, which can be solved by mathematical programming methods. Besides, the proposed formulation is efficient to establish kinematical, equilibrium and mixed principles in the solid mechanics.

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