

# Multiphase Flow Simulation in Heterogeneous Porous Media using a Hybrid FVM-FEM Scheme

E. WENDLAND<sup>1</sup>, Departamento de Hidráulica e Saneamento, EESC-USP, C.P. 359, 13560-970, São Carlos - SP, Brasil

D. FLENSBERG<sup>2</sup>, Westfälische Landesbank, WestLB, Herzogstr. 15, D-40217, Düsseldorf, Germany.

**Abstract.** This work describes a numerical scheme for simulation of two-phase flow in heterogeneous porous media. The system of governing equations is transformed into an equivalent problem, as proposed by Peaceman (1977). For the numerical solution, the operator split technique is used, separating the advection-diffusion equation in a hyperbolic and a parabolic part. The Finite Element and Finite Volume Methods are used for the spatial discretization of the problem. The reliability of the scheme is demonstrated for a typical problem of Petroleum Engineering, showing better performance than a commercial software.

## 1. Introduction

Mathematical modelling and numerical simulation of flow in the subsurface depends strongly on the composition and characteristics of the transported fluids. Different interfacial tension, viscosity and density determine the hydrophobic behaviour of some fluids. Immiscible fluids present in porous media a different behaviour than water and their flow needs to be specifically described. On the one hand, the subject matter addresses environmental problems dealing with protection or remediation of aquifers contaminated by hydrocarbons (gasoline, oil or BTX - Benzol, Toluol, Xylol). On the other hand, the problem of fossil fuel extraction (petroleum) also requires the analysis of immiscible fluid flow in the subsurface.

The system of differential equations that describe the process is strongly non-linear. Mass conservation equations for each phase involved are coupled through secondary conditions. Furthermore, the capillary pressure and the relative permeability depend on the saturation of the involved phases, which are also unknowns in the problem. In presence of heterogeneous porous media these effects are amplified due to the specific behaviour of the considered phases. Traditionally, the solution of the system of equations can be grouped in one of the methods discussed by Aziz

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<sup>1</sup>ew@sc.usp.br

<sup>2</sup>daniel-flensberg@westlb.de

and Settari (1983): SS (*Simultaneous Solution*), IMPES (*IMplicit Pressure, EXplicit Saturation*) or SEQ (*SEQuential solution*). Dorgarten (1989) and Helmig (1993) presented overviews about possible approximations and applications. Most works differentiate from each other by the unknowns chosen for the solution.

In this work a numerical scheme for the simulation of multiphase flow in heterogeneous porous media is presented. The system of partial differential equations resulting from the mass balance for different phases is transformed in an equivalent problem, as proposed by Peaceman (1977). For the numerical solution, the operator-split technique is used, allowing a combination of the Finite Element Method (FEM) and Finite Volume Method (FVM) for the spatial discretization of the problem. In section 2 below, the fundamentals of the mathematical formulation are presented. Section 3 presents a practical problem of Petroleum Engineering, through which the reliability of the scheme is demonstrated in comparison with results of a commercial software. The conclusions of the work are summarized in section 4.

## 2. Mathematical Model

The present model is restricted to two-phase flow in isothermal regime. Flow in porous media is mathematically described by mass conservation equations in a control volume. The partial differential equations can be formulated for both wetting (subscript  $w$ ) and non-wetting (subscript  $n$ ) phases as a system of equations with the pressure and saturation as primary unknowns:

$$\frac{\partial (\phi S_w \rho_w)}{\partial t} - \nabla [\rho_w \lambda_w \mathbf{K} \nabla (p_w + \rho_w g Z)] - q_w \rho_w = 0, \quad (2.1)$$

$$\frac{\partial (\phi S_n \rho_n)}{\partial t} - \nabla [\rho_n \lambda_n \mathbf{K} \nabla (p_n + \rho_n g Z)] - q_n \rho_n = 0, \quad (2.2)$$

where  $\phi$  [-] is the soil porosity,  $S$ [-], the phase relative saturation,  $\rho$  [kg/m<sup>3</sup>], the fluid density,  $t$  [s], the time,  $\lambda$  [1/(Pa.s)], the phase mobility,  $\mathbf{K}$  [m<sup>2</sup>], the absolute soil permeability,  $p$  [Pa], the pressure of each phase,  $g$  [m/s<sup>2</sup>], the gravity constant,  $Z$  [m], the vertical position and  $q$  [1/s] is a fluid source. In this model, only the phase mobility ( $\lambda$ ) and the pressure of each phase ( $p$ ) are variables, which depend on the respective saturation ( $S$ ) of the phases  $w$  and  $n$ . The remaining soil characteristics can be seen as constants.

The mathematical description of the physical process is completed by secondary conditions:

$$S_n = 1 - S_w, \quad (2.3)$$

$$p_n = p_c + p_w. \quad (2.4)$$

The variable  $p_c$  represents the capillary pressure, which is responsible for a meniscus appearing between two immiscible phases. This term can be defined as a function of the saturation of the wetting phase

$$p_c = f(S_w). \quad (2.5)$$

Equations (2.1) and (2.2) together with the secondary conditions (2.3) and (2.4) represent a coupled system of four differential equations for the determination of primary unknowns  $p_w, p_n, S_w$  and  $S_n$ . The definition of initial and boundary conditions complete the formulation of the model.

### 2.1. Transformed model for two-phase flow

Generally, the simultaneous solution (SS) of the traditional formulation (Eqs. (2.1) and (2.2)) is based on the definition of two principal variables (e.g.  $p_n, S_w$ ) and the elimination of the other two (e.g.  $p_w, S_n$ ) through the incorporation of the secondary conditions (2.3) and (2.4). According to Flensburg (1997), this approximation consists of a system of differential equations with singular perturbation in the time derivative. In problems with low compressibility, this characteristic leads to the propagation of round-off errors during the numerical solution. This behaviour can be alleviated transforming the original system in an equivalent one, as proposed by Peaceman (1977). The transformation consists in the introduction of a mean pressure ( $p$ ), which substitutes adequately both pressures (one for each phase) in the governing equations. The objective of this transformation is to remove the time derivative of the saturation. The equivalent form for a domain  $\Omega$  delimited by a boundary  $\Gamma$  is given by the system of equations

$$\phi \frac{\partial S_w}{\partial t} + \nabla(\mathbf{u}_w) = q_w, \tag{2.6}$$

where

$$\mathbf{u}_w = \mathbf{u}_w^{adv}(S_w, \mathbf{u}) + \mathbf{u}_w^{dif}(S_w),$$

$$\mathbf{u}_w^{dif}(S_w) = \mathbf{K} \frac{\lambda_n \lambda_w}{\lambda} \nabla p_c, \quad \text{the diffusive term,}$$

$$\mathbf{u}_w^{adv}(S_w, \mathbf{u}) = \frac{\lambda_w}{\lambda} \mathbf{u} + \mathbf{K} \frac{\lambda_n \lambda_w}{\lambda} (\rho_n - \rho_w) g \nabla Z, \quad \text{the advective term, and}$$

$$\phi c_t \frac{\partial p}{\partial t} + \nabla(\mathbf{u}) = q_t \tag{2.7}$$

where

$$c_t = \frac{1}{\phi} \frac{\partial \phi}{\partial p} + S_w c_w + S_n c_n, \quad \text{the total compressibility of the system,}$$

$$p = \frac{p_w + p_n}{2}, \quad \text{the mean pressure,}$$

$$\mathbf{u} = \mathbf{K} \lambda \left[ -\nabla p + \left( \frac{\lambda_w - \lambda_n}{2} \right) \nabla p_c + \bar{\rho} g \nabla Z \right], \quad \text{the total flow velocity,}$$

$$\lambda = \lambda_w + \lambda_n, \quad \text{the total mobility of the two-phase system,}$$

$$\bar{\rho} = \frac{\lambda_w \rho_w + \lambda_n \rho_n}{\lambda}, \quad \text{a simplification and}$$

$$q_t = q_w + q_n, \quad \text{the total source rate.}$$

After this transformation, the equations are denominated as saturation equation (2.6) and pressure equation (2.7). The primary unknowns of the model are the mean pressure  $p$  and the saturation of the wetting phase  $S_w$ . Both equations are coupled since the total velocity of the system  $\mathbf{u}$ , which depends on the pressure, appears in the saturation equation, and the terms of the pressure equation also depends on the saturation of the wetting phase ( $S_w$ ).

## 2.2. Time discretization

For the time approximation, an alternative scheme, according to Flensburg (1997), is used. In order to treat the diffusive and the advective terms of the saturation equation with different schemes, the operator-split technique (Marchuk, 1995) is applied. The procedure consists of the separation of the advection-diffusion equation (2.6) in two parts. At each time step, the saturation equation is approximated by a hyperbolic equation

$$\phi \frac{\partial S_w}{\partial t} + \nabla \mathbf{u}_w^{adv}(S_w, \mathbf{u}) = q_w, \quad (2.8)$$

followed by a parabolic one

$$\phi \frac{\partial S_w}{\partial t} + \nabla \mathbf{u}_w^{dif}(S_w) = 0. \quad (2.9)$$

The hyperbolic part (2.8) is approximated by an explicit scheme. As a result, the high frequency components of the solution are not damped out and the pressure and saturation equations are numerically separated. For the parabolic equation (2.9), an implicit scheme is used in order to avoid stability criteria. The implicit approximation of the parabolic part does not couple both equations because the term  $\mathbf{u}_w^{dif}$  does not depend on the total velocity, which in turn is calculated in the pressure equation.

After these preparations the algorithm for a time stepping scheme can be presented. Let  $S_w^0$ ,  $u^0$  and  $p^0$  be values at time  $t = 0$ . A time step  $\Delta t$  and an end time  $T_{end} := N \cdot \Delta t$  are chosen and the index of the time step is assumed as  $n := 0$ .

1. Determine  $S_w^{n+1/2}$  from

$$\phi \frac{S_w^{n+1/2} - S_w^n}{\Delta t} + \nabla \cdot u_w^{adv}(S_w^n, u^n) = q_w.$$

2.  $S_w^{n+1}$  is determined from

$$\phi \frac{S_w^{n+1} - S_w^{n+1/2}}{\Delta t} + \nabla \cdot u_w^{dif}(S_w^{n+1}) = 0.$$

3. Determine  $u^{n+1}$  and  $p^{n+1}$  from equation (2.7) in a mixed formulation

$$[K\lambda(S_w^{n+1})]^{-1} u^{n+1} + \nabla p^{n+1} = q_c(S_w^{n+1}) + q_g(S_w^{n+1}),$$

$$\nabla \cdot u^{n+1} + \phi c_t \frac{p^{n+1}}{\Delta t} = \phi c_t \frac{p^n}{\Delta t} + q_t,$$

where  $q_c = (\frac{\lambda_w - \lambda_n}{2}) \nabla p_c$  and  $q_g = \bar{\rho} g \nabla Z$ .

4. If  $n = N - 1$ , terminate, otherwise replace  $n$  by  $n + 1$  and return to step 1.

### 3. Practical Application

A prototype was developed representing a typical problem of Petroleum Engineering. The model consists of the simulation of an enhanced oil recovery process known as “five-spot water flood”. Here an initially oil saturated reservoir is drained through water injection. In this example, the drainage process is simulated without capillarity and gravity effects, consisting in a Buckley-Leverett type problem (advection dominated).

This model was chosen in order to verify the dependency of the method on the mesh discretization, especially in cases with spatial heterogeneity. According to the classical problem described by Spivak et al. (1977), a regular mesh will be used. As shown in Figure 1, the wetting phase (water) is injected in the lower left corner by an injection well and the non-wetting phase (oil) is drained in the upper right corner by a production well. The flow direction is, in principle, diagonal to the mesh orientation.



than in zone 1, constituting an obstacle to the flux. With this problem, we intend to investigate the behaviour of the presented method in a heterogeneous medium. The influence of the heterogeneity was investigated for different permeability ratios ( $K_1/K_2$ ), with values of 1000, 10 and 2.

The following physical properties were assigned to the model:

- porous media	porosity	$\phi = 0.2$
	permeability	$K_1 = 1.0 \cdot 10^{-7} \text{ m}^2$
- fluids (water and oil)	density	$\rho_\alpha = 1000 \text{ kg/m}^3$
	viscosity	$\mu_\alpha = 0.001 \text{ Pa.s}$

Relative permeabilities for both region are described according to Todd's model, given by the functions

$$\begin{aligned} k_{rw}(S_w) &= S_w^2 && \text{for the wetting phase and} \\ k_{rn}(S_w) &= (1 - S_w)^2 && \text{for the non-wetting phase.} \end{aligned}$$

Following initial and boundary condition were defined:

- saturation equation (convective part):

$$\begin{aligned} S_w(x, 0) &= 0.0 && \text{for } x \in \Omega, \\ S_w &= 1.0 && \text{in } \Gamma_{in} \end{aligned}$$

- pressure equation:

$$\begin{aligned} u \cdot n &= 0.0 && \text{in } \Gamma_{wall}, \\ u \cdot n &= -3.210^{-6} \text{ [m/s]} && \text{in } \Gamma_{in}, \\ u \cdot n &= 3.210^{-6} \text{ [m/s]} && \text{in } \Gamma_{out} \end{aligned}$$

For the pressure equation it is necessary to define at least one Dirichlet condition. Considering the symmetry condition, a pressure is defined for the lower left corner:

$$p = p_w = p_n = 10^5 \text{ [Pa]} \text{ in } P(0, 0).$$

### 3.1. Numerical results

A regular mesh discretized the domain with 32 divisions (9.375m) in the  $x$  and  $y$  directions, totalling 1089 nodes and 1024 square elements. The problem was simulated up to 800 days, with time intervals ( $\Delta t$ ) of 1 day.

For the interpretation of the results in the figures below, the caption of Figure 2 should be observed. The grey tones of the isosurfaces are related to the percent water saturation in the porous media at the end of the simulation period.

Figure 3 shows isosurfaces of water saturation after 800 days of injection for the five-spot water flood model with permeability ratio ( $K_1/K_2$ ) equal to 1000. This example represents an extremely heterogeneous case. As it can be observed, oil is displaced by water injected in the lower left corner. The displacement does not occur in zone 2 due to its low permeability value. Zone 1 constitutes the preferential flow path.



Figure 2: Caption: Water saturation (in %) in the heterogeneous porous media.

The numerical solution presents a sharp wetting front as expected in Buckley-Leverett type problems (dominated by advection, without capillarity and gravity effects). The water saturation changes from 0% to 70% in a distance corresponding to two elements. In the interface between zones 1 and 2, problems of oscillation and numerical diffusion are not observed. The saturation varies from 0% to 70% over a length of a single element in this region.

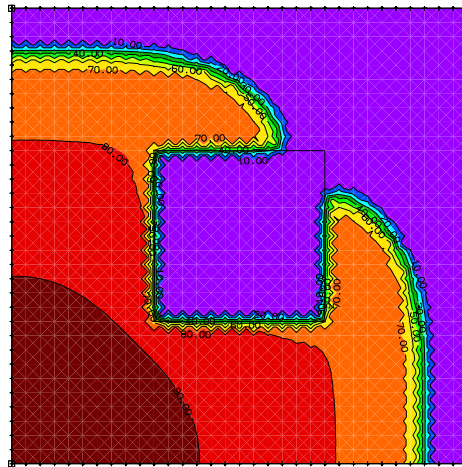


Figure 3: Water saturation after 800 days of injection with permeability ratio  $K_1/K_2 = 1000$ .

This result was obtained obeying the CFL (Courant-Friedrich-Levy) condition, which restricts the time step discretization ( $\Delta t$ ). The time step was chosen for the element with the worst condition. For the element that contains the lower left corner  $P(0,0)$ , the maximum flow velocity is expected. Ignoring the numerical errors in this element, the process can be simulated with larger time steps.

In Figure 4a, the result for a permeability ratio ( $K_1/K_2$ ) of 10 is presented. Due to the lower permeability difference, the oil displacement can also be observed in zone 2. As a consequence, the advancement of the saturation front (10% surface) in the production well direction is delayed in comparison with Figure 3. This indicates the observance of a mass conservation in the simulation. The interface between zones 1 and 2, represented by a continuous line in the figure, does not represent a

barrier to the oil displacement anymore. The variation of the saturation from 0% to 70% is still observed over the length corresponding to two elements.

The results obtained with permeability ratio ( $K_1/K_2$ ) equal to 2 are shown in Figure 4b. The saturation distribution is equivalent to the last case, but with less influence of the heterogeneity. The water invasion in zone 2 (lower permeability) is almost independent of the heterogeneity. The advance of the saturation front (10% surface) in the production well direction resembles the homogeneous case solution (Spivak et al., 1977). The saturation variation from 0% to 70% is sharp as in the previous cases.

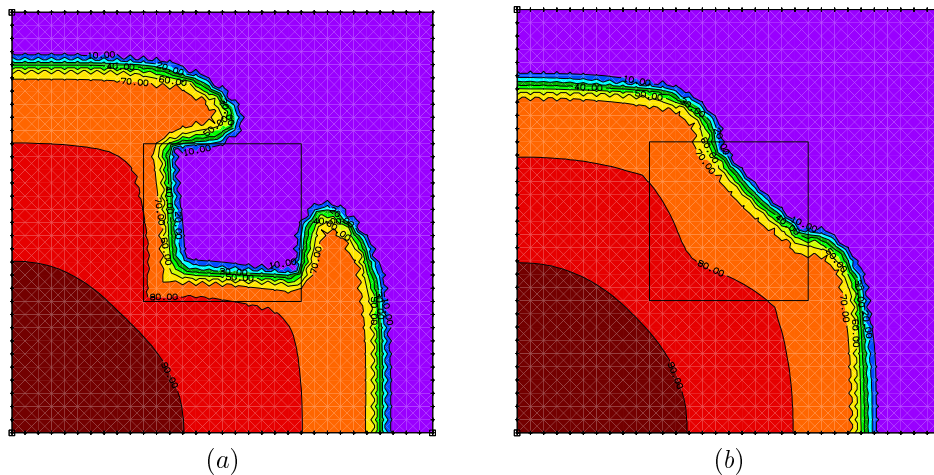


Figure 4: Water saturation after 800 days of injection with permeability ratio (a)  $K_1/K_2 = 10$ ; (b)  $K_1/K_2 = 2$ .

In order to validate the solution, the first case ( $K_1/K_2=1000$ ) was also simulated with the commercial software IMEX (CMG, 1999), widely utilised in the petroleum industry. Isolines of 10% to 90% of water saturation obtained with the IMEX package (Figure 5a) are shown in comparison with the result of the proposed method (Figure 5b). The commercial simulator shows a larger spread of isolines, due to the numerical diffusion inherent to schemes based on the finite difference method. In the advancing front, the water saturation varies from 0% to 70% over a length of four elements. One can conclude that the proposed method (lower numerical diffusion) reproduces the Buckley-Leverett problem (advection dominated) more exactly.

## 4. Conclusion

A hybrid formulation based on the finite element method (FEM) and finite volume method (FVM) for the simulation of multi-phase flow in heterogeneous porous media has been presented. For verification of the proposed algorithm, a typical case of Petroleum Engineering (five-spot water flood) was chosen. This Buckley-Leverett



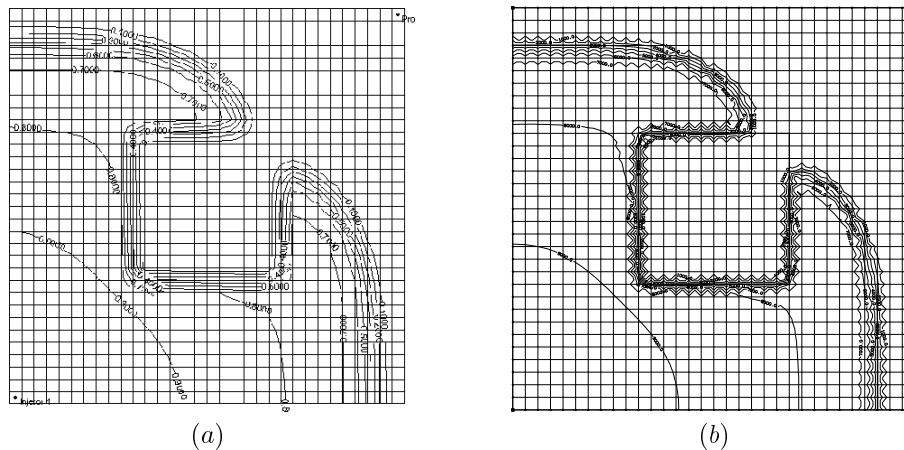


Figure 5: Water saturation (10% to 90%) computed with (a) commercial simulator; (b) proposed method.

problem (advection dominated, without capillarity and gravity effects) generally imposes numerical difficulties to the most approximation schemes due to the presence of the advective term. The results shown in Figures 3 to 5 clearly demonstrate the efficiency of the method. Even when strong heterogeneities are present, the numerical solution shows a sufficient sharp advancing front. In the interface between both permeability zones, oscillation and numerical diffusion do not represent a difficulty. In comparison with commercial software, the proposed scheme reproduced the physical problem more faithfully.

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**Resumo.** Este trabalho descreve um esquema numérico destinado à simulação de escoamento bi-fásico em meios porosos heterogêneos. O sistema de EDPs resultante do balanço de massa para as diferentes fases é transformado em um problema equivalente, conforme proposto por Peaceman (1977). Para a solução numérica do problema, a técnica de operator-split é utilizada. A idéia do operator-split consiste em separar a equação de advecção e difusão em uma parte hiperbólica e outra parabólica. A parte hiperbólica é aproximada por um esquema explícito, de forma que as componentes de alta frequência da solução não sejam amortecidas e as equações da pressão e da saturação sejam numericamente desacopladas. Para a equação parabólica, um esquema implícito é utilizado, de modo a contornar os critérios de estabilidade. Uma combinação dos Métodos de Volumes Finitos e Elementos Finitos é utilizada para a aproximação espacial do problema. A viabilidade do esquema desenvolvido é demonstrada para um problema típico da Engenharia de Petróleo.

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