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On a model for the fake news diffusion between two interacting populations

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ABSTRACT. The dynamics of information propagating among populations that interact might have an enormous impact on public opinion, particularly when such information is false, known as fake news. In this contribution, we propose and analyze the fake news dissemination that occurs when two distinct sub-populations (not necessarily homogeneous) share information, using a reinterpretation of a compartmental model for disease dissemination. We show the model's well-posedness. Furthermore, we utilize the model solution property to derive an estimation that allows one to estimate the impact of the influence of one population on the other in the fake news dissemination. The theoretical results are complemented with numerically simulated scenarios for the dynamics of fake news spreading among populations, with the model parameters associated with some human development and influence indices among countries. The results obtained show that the speed of diffusion of fake news among populations is largely impacted by the gap between the human development indices of each population and the influence of one population on another. It is also shown that a small percentage of control over information shared by the population leads to a large decrease in the amount and velocity of fake news diffusion.

Keywords: fake news modeling, multi-population, diffusion dynamics, information.

1 INTRODUCTION

The spread of invented or misleading information, nowadays known as fake news, has always been the subject of human behavior [3, 11, 19, 20, 23]. This information is normally produced and distributed with the proposal of getting some sort of advantage, for example, to profit from the number of visitors to the published article on-line or to discredit a person (usually a political opponent in an election) in order to conduct a public opinion poll [3, 10, 11, 19, 20, 23]. In [21] it was shown that fake news normally circulates more widely, faster and with 70% more chances

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of being shared than truth. It gains another dimension with the advent of social media [3, 10, 11, 19, 20, 23].

Given the amount and weight of fake news that circulates nowadays, the importance of having information on its dynamics and the mechanisms to limit its advances and effects in our society becomes evident [3, 10, 11, 19, 20, 23]. It makes the truthfulness of the information we receive online one of our biggest concerns. Modeling the capability of detecting and the dynamics of spreading false information has received attention recently in [4,6,7,8,13] and references therein.

Since spreading fake news acts as an epidemic [3,10,11,19,20,23], the authors in [4,6,8] propose a modification of SIR-type compartmental models to spread diseases [1] to model the spread of fake news. Using a linearization around the initial conditions and parameters that represent social development, [4, 6] associated the stiffness ratio of the SIR dynamics with the velocity of the spread of fake news in a population. They show numerically that the larger the stiffness ratio, the faster the truth is restored.

In this contribution, we follow the ideas in [4, 6] and reinterpret the uses of a compartmental model to spread diseases between two populations (here assumed as countries) as the dynamics of the spread of fake news between two populations that can share information. We extend the analysis presented by the authors in [18], adding a derived estimate that shows the influence of one population on the other on the spread of fake news. We also present more simulated scenarios using the human development index, the internet penetration index, and the FBIC[1] index [15], a measurement of the influence of one country on another as the model parameter in the spread of fake news.

Article organization and novelties: In Section 2, we describe the proposed model and its well-posedness. In Section 3, we derive a simple estimate that shows the influence of population interaction on the diffusion of fake news. In Section 4, we analyze some of the capabilities of the proposed model with parameters that reflect some indicators of human development and indices that measure the influences of one country on another. The simulated scenarios presented in Subsection 4.1 show that information sharing among populations can have a large impact on the dynamics of the spread of fake news. We also show numerically that a small amount of control in the model leads to a large decrease in the amount and velocity of fake news diffusion in Subsection 4.2. Furthermore, in Section 4.3, we show that the heterogeneity of the human development index between populations affects the speed of the spread of fake news, but the inverse of the stiffness ratio is not monotonic, affecting the speed of the spread of fake news, contrary to the conclusions in [4, 6]. In Section 5, we summarize the conclusions and address future directions.

[1] How the FBIC index is calculate can be see at https://korbel.du.edu/fbic or in [15].

2 THE PROPOSED DYNAMICS FOR FAKE NEWS SPREADING AMONG TWO INTERACTING POPULATIONS

Since fake news spreads like a virus [10], we use an alternative interpretation of the dynamics of the compartmental SIR model [1], to describe the diffusion of fake news in a scenario composed of two distinct populations of individuals (say P_1 and P_2) who share information. This reinterpretation consists of assuming that each population P_i, is proportionally subdivided into compartments of individuals $S_i(t)$ that are susceptible to believing in fake news shared by individuals in the compartment $C_i(t)$ who are already convinced that the fake news is true and shear this information with other individuals in the population at some rates β_{ij}, and individuals that could reestablish the truth of the information at a rate of γ_i, denoted by $R_i(t)$, after having been in the compartment $C_i(t)$, for $i = 1,2$ at any time $t \geq 0$. We will also analyze the possibility of a pulse control strategy to restore truth in the population j, for $j = 1,2$, respectively. Such strategies are implemented or not in a population according to the choice of the parameter $\xi_j \in \{0,1\}$. Therefore, if $\xi_j = 1$ then there exists (in the case of $\xi_j = 0$ there is no such control) a mechanism that acts to reestablish the truth, with the efficacy of ρ_j, in the population j. Using the mass and action laws [1], it is possible to argue that the dynamics shall follow the coupled system of differential equations

$$
\begin{aligned}
\dot{S}_j(t) &= S_j(t)(-\beta_{jj}C_j(t) - \beta_{ij}C_i(t)) - \xi_j\rho_j S_j(t) \\
\dot{C}_j(t) &= S_j(t)(\beta_{jj}C_j(t) + \beta_{ij}C_i(t)) - \gamma_j C_j(t) \\
\dot{R}_j(t) &= \gamma_j C_j(t) + \xi_j\rho_j S_j(t)
\end{aligned}
\tag{2.1}
$$

and initial conditions

$$
S_j(0) = P_j - C_j(0) \geq 0, C_j(0) \geq 0, R_j(0) \geq 0\,, \tag{2.2}
$$

for $i, j = 1,2$ and $i = 3 - j$. All parameters in model (2.1) are assumed to be nonnegative and constant. The model represented by (2.1) follows the traditional compartmental approach used in mathematical epidemiology, as described in [2]. It is specifically an SIR-type model that includes aspects of vaccination and interaction between two distinct host populations [5]. In this context, ξ_j indicates the presence of a vaccine, while ρ_j is associated with the effectiveness of the vaccination, as explained in [2].

Here we follow the interpretation of the parameters given in [6, 7] and assume that β_{jj} and γ_j are associated with the economic index of development of a population, the *internet penetration index* IPI_j and the *human development index* HDI_j, of each population $j = 1,2$. In particular, $\beta_{jj} = \sigma_{jj} IPI_j$ and $\gamma_j = \alpha_j HDI_j$, for $j = 1,2$, where the proportions are $\sigma_{jj}, \alpha_j \in]0,1]$. In general, $\sigma_{jj} > \alpha_j$ since it is easier to spread a lie than to reaffirm the truth [21]. The parameter β_{ij} is related to the influence of population j on population i, for $i, j = 1,2$ and $i = 3 - j$. In this contribution, we discuss the case where $\beta_{ij} = FBIC_{ij}$, where $FBIC_{ij}$ is the index that consists of a bilateral measure of the influences of one country on another [15].

2.1 Well-posedness

In this subsection, we briefly discuss the well-posedness of a solution for the model (2.1) with initial conditions (2.2) and some of its properties that will be used in the forthcoming analysis of fake news spreading.

Lemma 1. *Let $P(t) = P_1(t) + P_2(t)$, where $P_j(t)$ is the total of individuals in population $j = 1,2$. Then $P(t)$ is constant for any $t \geq 0$.*

Proof. Summing up the two sides of (2.1), we arrive at the conclusion that $\dot{P}(t) = 0$, from which the assertion follows. □

Lemma 2. *If a solution $U(t) = [S_1(t), C_1(t), R_1(t), S_2(t), C_2(t), R_2(t)]^T$ of* (2.1) *with initial conditions* (2.2) *exists, then it is uniformly bounded by $P(0)$. In particular, all the coordinates of $U(t)$ are uniformly bounded.*

Proof. Let $\|\cdot\|_1$ be $1-norm$ in $\mathbb{R}^n$. It follows that $\|U(t)\|_1 \leq \|P(t)\|_1$ for any $t \geq 0$. Since $P(t)$ is constant (see Lemma 1), the assertion follows. □

Proposition 3. *Let the map $F(t,U(t))$ define the vector field on the right-hand side of the model* (2.1). *Then:*

i) *$F(t,U(t))$ is continuous at any $t \geq 0$.*

ii) *There exist constants w_1 and w_2 such that $\|F(t,U(t))\| \leq w_1 + w_2\|U(t)\|$.*

iii) *$F(t,\cdot)$ is Lipschitz continuous whit respect to the second coordinate.*

Proof. The assertion in Item i) follows directly from the fact that each coordinate of $F(t,U(t))$ is given by the sum and product of continuous functions and constant parameters.

The item ii) follows immediately from the definition of $F(t,U(t))$ and the boundedness of each coordinate of $U(t)$ given by Lemma 1.

A direct calculation shows that the Jacobian matrix of the system (2.1) is given by

$$JF(t,U(t)) = \begin{bmatrix} a_{11} & -\beta_{11}S_1(t) & 0 & 0 & -\beta_{21}S_2(t) & 0 \\ -a_{11} - \xi_1\rho_1 & \beta_{11}S_1(t) - \gamma_1 & 0 & 0 & \beta_{21}S_2(t) & 0 \\ \xi_1\rho_1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{12}S_1(t) & 0 & a_{44} & -\beta_{22}S_2(t) & 0 \\ 0 & \beta_{12}S_2(t) & 0 & -a_{44} - \xi_2\rho_2 & \beta_{22}S_2(t) - \gamma_2 & 0 \\ 0 & 0 & 0 & \xi_2\rho_2 & \gamma_2 & 0 \end{bmatrix}$$

where $a_{11} = (-\beta_{11}C_1(t) - \beta_{21}C_2(t)) - \xi_1\rho_1$ and $a_{44} = (-\beta_{22}C_2(t) - \beta_{12}C_1(t)) - \xi_2\rho_2$. Therefore, it follows from Lemma 1 that there exists a constant $L > 0$ such that $\||\|JF(t,U(t))\||\|$ uniformly in $t \geq 0$. Hence, the mean value theorem implies that

$$\||\|F(t,U(t)) - F(t,\tilde{U}(t))\| \leq L\|U(t) - \tilde{U}(t))\|, \qquad (2.3)$$

concluding assertion iii). □

The following theorem is one of the main theoretical results of this contribution.

Theorem 4. *Let the general assumptions regarding the model* (2.1) *satisfied, with initial conditions* (2.2). *Then, there exists a unique continuous differentiable and nonnegative solution* $U(t;\xi_j) := U(t) = (S_1(t), C_1(t), R_1(t), S_2(t), C_2(t), R_2(t))^T \in \mathbb{R}^6$ *for any* $t \geq 0$*, for any choice* $\xi_j \in \{0,1\}$*. The solution* $U(t;\xi_j)$ *continuously depends on the model parameters and initial conditions.* **Proof.** Given the results in Proposition 3, the theorem statements follow from the standard result of well-posedness for systems of initial value problems, e.g., [17]. □

3 THE EFFECT OF POPULATION INTERACTION ON THE SPREAD OF FAKE NEWS

In the following, we explore the effects of the interaction between the populations on the spreading of the fake news with no control (i.e. $\xi_j = 0$, for $j = 1,2$).

Since the solution $U(t)$ has all its components non-negative (see Theorem 4, we see from the model (2.1) that $S_j(t)$ decreases for $j = 1,2$.

Adding the equations for S_j and C_j in the model (2.1), for $j = 1,2$ we find that is satisfies the conservation law

$$S_j(t) + C_j(t) + \gamma_j \int_0^t C_j(s)ds = S_j(0) + C_j(0)\,. \tag{3.1}$$

It follows from (3.1) that $C_j(t) < \infty$, for all $t \geq 0$. Also, $\int_0^\infty C_j(t)dt \leq \infty$. Moreover, (3.1) implies in $\lim\limits_{t\to\infty} C_j(t) = 0$. Therefore, the spread of false news in the population is such that $C_j(t)$ starts to increase and then decreases. Since $C_j(t)$ is smooth (see Theorem 4), we conclude that its trajectory has a concave hump that starts in $C_j(0) \geq 0$ and ends in $C_j(\infty) = 0$. Consequently , $C_j(t)$ attains a maximum at the turning point t_p^j, within $C_j(t_p^j) \neq 0$ and $\dot{C}_j(t_p^j) = 0$. Hence, it follows from (2.1) that

$$S_j(t_p^j) = \frac{\gamma_j}{\beta_{jj}} \left(\frac{1}{1 + \frac{\beta_{ij}}{\beta_{jj}} \frac{C_i(t_p^j)}{C_j(t_p^j)}} \right), \quad \text{for } i = 3-j, \quad i,j \in \{1,2\}\,. \tag{3.2}$$

Since the parameter β_{ij} represents the influence of population i on population j, it follows that the analysis of equation (3.2), reveals the influence of population interaction on the spread of fake news. This is the content of the following remark.

Remark 1. *Consider the identity* (3.2) *derived above. Then:*

1. *The turning point* t_p^j *depends on the parameters of the model, as well as on the initial conditions* (2.2). *Therefore,* (3.2) *also depends on such conditions. See Figure 2 below for the verification of such a dependence.*

2. *If the population i does not influence the population j, that means $\beta_{ij} = 0$, then, it follows from* (3.2) *that*
$$S_j(t_p^j) = \frac{\gamma_j}{\beta_{jj}}.$$
In this case, the populations are isolated. This means that if fake news begins in the population j, then the population i will remain free of fake news for $i = 2 - j$, and $j \in \{1,2\}$. See Figure 1 (f).

3. *The largest is the influence of one population on the other that leads the largest number of people to believe in fake news. Indeed, since the quantity corresponding to β_{ij} appears in the denominator in* (3.2)*, we have that $S_j(t_p^j)$ decreasing as β_{ij} increases. Therefore, from* (3.1)*, $C_j(t_p^j)$ increases. See Figure 1 (a).*

4 NUMERICAL SIMULATED SCENARIOS

The simulations presented in this contribution, IVP (2.1)-(2.2) are solved using the Euler forward method with step size $h = 10^{-3}$, in the time interval $[0,1000]$, implemented in Python 3.9. The population is normalized, meaning that $P_j^k = 1$ for any $j = 1,2$ and $k = 1,\cdots,6$, with P_j^k representing different countries: P_j^1 from Brazil, P_j^2 from France, P_j^3 from India, P_j^4 from Mozambique, P_j^5 from the United States and P_j^6 from South Sudan. These choices were made because they characterize a distinct variety of socially developing populations.

In this paper, we follow the interpretation of the parameters given in [6,7] and assume that $\beta_{jj} = \sigma_{jj} IPI_j$ and $\gamma_j \alpha_j HDI_j$ are associated with the economic index of development of a population, where IPI_j is the Internet penetration index, a percentage metric that measures the extent to which the internet is accessible and used by a population [22], and HDI_j is the human development index[2], that is metric that relates to the economic and social progress of a population [16]. Here we also follows the authors in [6,7] an uses $\sigma_{jj} = 1/10$ and $\alpha_j = 1/100$, since it is estimate that is 10-times easier to to spread a lie than to reaffirm the truth [21]. The corresponding values of the parameters in the model (2.1) are specified in Table 1.

The primary distinction between our methodology and that presented in [6,7] for modeling the dissemination of fake news lies in the interaction of two distinct populations or countries. Therefore, we propose to interpret the parameter β_{ij} as the influence of population j on population i, for $i, j = 1,2$ and $i = 3 - j$. In this contribution, we propose using $\beta_{ij} = p_{ij} M_j^k / 10$ as a measure of the influence of a population (country) i in the population (country) j calculated according to FBIC[3] index [9,15] and presented in Table 2, where M_j^k is the maximum influence of the popula-

[2]The HDI is a composite statistic used to rank countries based on human development levels, considering factors like life expectancy, education, and per capita income. It is used alongside other indices, such as the internet penetration index (IPI), to provide a comprehensive analysis of a population's development status.

[3]The Foreign Bilateral Influence Capacity (FBIC) Index is built upon the idea that two main factors affect the ability of states to exert influence in the international system. First, the degree of interaction across economic, political, and security dimensions creates opportunities for states to influence each other. Second, the relative dependence of one state on another for crucial aspects of economic prosperity or security creates opportunities for the more dominant state to cause the more dependent state to make decisions that they would not have otherwise made [15].

tion j onto the population i (here represented as the maximum influence of the analyzed countries k, for $k = 1, \cdots, 6$, as the population j), since 1960, given by the last column of Table 2, and p_{ij} corresponds to the influences of a population (country) i in the population (country) j, for year 2019, given by the position ij in Table 2. For a complete overview of how p_{ij} is constructed, and how to obtain M_j^k, please see [9, 15] and the references therein.

Table 1: Values of the parameters β_{jj} and γ_j for the population P_j^k, for $j = 1, 2$, where $k = 1, \cdots, 6$ is the corresponding country.

	Brazil P_j^1	France P_j^2	India P_j^3	Mozambique P_j^4	US P_j^5	S. Sudan P_j^6
β_{jj}	0.072	0.089	0.035	0.021	0.075	0.009
γ_j	0.008	0.009	0.006	0.005	0.009	0.004

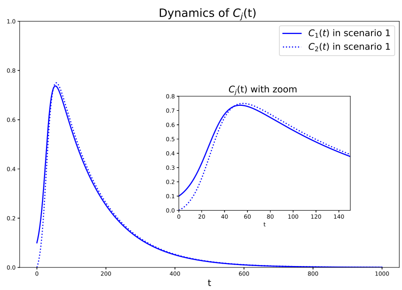
Table 2: Values used for calculate the parameters β_{ij} and M_j^k for the population $i, j = 1, 2$ with $i = 3 - j$ in the simulated scenarios, where $k = 1, \cdots, 6$ is the corresponding country. Values are obtained from *https://korbel.du.edu/fbic*.

	Brazil P_i^1	France P_i^2	India P_i^3	Mozambique P_i^4	US P_i^5	S. Sudan P_i^6	M_j^k
P_j^1	-	0.0218	0.0246	0.0409	0.0464	0	0.3946
P_j^2	0.1455	-	0.1115	0.0146	0.1248	0.0002	0.7570
P_j^3	0.0318	0.0373	-	0.0826	0.0552	0.0124	0.2699
P_j^4	0.0005	0.0002	0.0046	-	0.0001	0	0.1112
P_j^5	0.2854	0.2534	0.2203	0.0510	-	0.0409	0.7567
P_j^6	0	0	0.00007	0	0.00001	-	0.04829

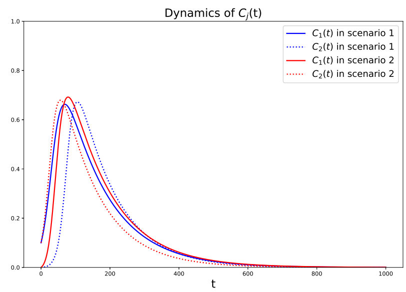
4.1 Influences of the parameters in the fake news dynamics

In this subsection, we present some numerical simulated scenarios with the aim of investigating the influences of the parameters on the fake news dynamics. In all the simulated scenarios presented in this subsection, there is no control, that is, $\xi_1 = \xi_2 = 0$.

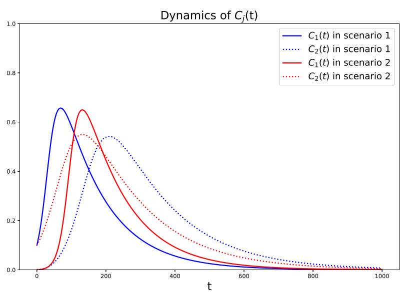
Simulated Scenario A: In this scenario we present the simulation between $P_1 := P_1^1$ for Brazil and $P_2 = P_2^k$ representing the countries of Brazil, France, India, Mozambique, United States, and South Sudan for $k = 1, \cdots, 6$, respectively. In Figure 1, two simulated situations are presented: In **scenario 1** the initial conditions (2.2) are such that $C_1(0) = 0.1$ and $C_2(0) = 0$, meaning that fake news begins to circulate between P_1 while P_2 is free from fake news circulating. In **scenario 2** the initial conditions (2.2) are $C_1(0) = 0$ and $C_2(0) = 0.1$, which means that the fake news begins to circulate between P_2 while P_1 is free from the fake news that circulates. The simulation parameters are those presented in Tables 1-2.



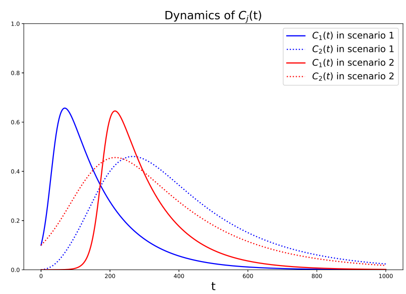
(a) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^1$ (Brazil) in **Scenario A**.



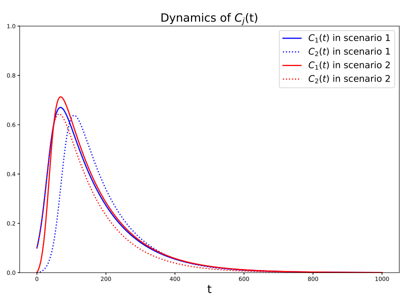
(b) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^2$ (France) in **Scenario A**.



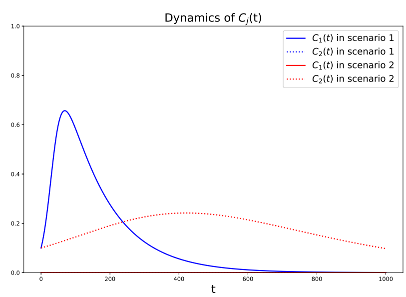
(c) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^3$ (India) in **Scenario A**.



(d) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^4$ (Mozambique) in **Scenario A**.



(e) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^5$ (United State) in **Scenario A**.



(f) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^6$ (South Sudan) in **Scenario A**.

Figure 1: Dynamics of C_j, for $j = 1,2$, from **Scenario A** with parameters in Table 1 and 2.

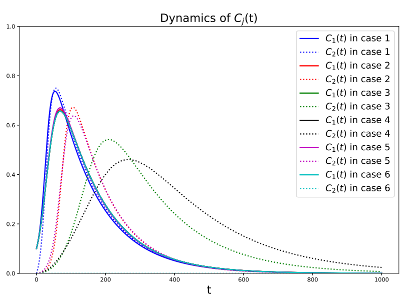
In Figure 1, we present the dynamics of the contaminated $C_j(t)$, for $j = 1,2$, of both populations in **Scenario A**, with the simulated scenarios 1 and 2 presented in the same Figures (a)-(f). In Figure 2 (a)-(b), we present the dynamics of C_j, for scenarios 1 and 2 in **Scenario A**, respectively.

In Figure 1 (a), both populations are Brazil ($P_1 = P_2$), therefore, we present only scenario 1 in this case. Moreover, the influence of Brazil on Brazil is 1, therefore, $\beta_{12} = \beta_{21} = M_j/10$. Therefore, the influence index is the highest among all simulated cases. Therefore, the dynamics of contamination achieves the highest peaks, compared to the other cases (see also Figure 2). This is consistent with Remark 1 iii). Figure 1 (a)-(f) and also Figure 2 (a)-(b) show that the largest is the influence of the population P_2^k, for $k = 1,\cdot,6$ the highest is the peaks in the dynamics in $C_j(t)$, which is also consistent with the identity (3.2).

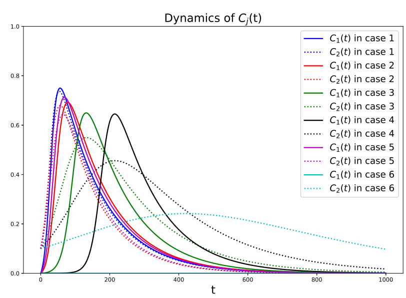
In Figure 1 (f), we have $\beta_{ij} = 0$ for $i, j = 1,2$ and $i = 3-j$. In these cases, the populations P^1 (Brazil) and P^6 (South Sudan) are isolated. Therefore, $C_2(t) = 0$ in scenario 1 and $C_1(t) = 0$ in scenario 2. This is consistent with Remark 1 ii).

Figure (2) shows the influence of the parameters and initial conditions on the delay of the turning point t_p^j. This is consistent with Remark 1 i).

An interesting phenomenon observed in Figure 1 (a)-(f) is that $C_1(t_p^1) \leq C_2(t_p^2)$ in scenario 2, while this is not observed in the scenario 1 in Figure 1 (b). In particular, this phenomenon is most prominent in populations with a lower development index (see Figure 1 (c)-(d). That means that Brazil (P_1) is the most influenced by fake news that starts to spread offshore. Such characteristics were not observed in the simulations presented in [18].



(a) $P_1^1 \times P_2^k$ with $k = 1,2,3,4,5,6$, in scenario 1 of **Scenario A**.



(b) $P_1^1 \times P_2^k$ with $k = 1,2,3,4,5,6$, in scenario 2 of **Scenario A**.

Figure 2: Simulated scenario for C_j in Scenario A. The case studies are as follows: Case 1: $P_1^1 \times P_2^1$ (Brazil) x (Brazil), Case 2: $P_1^1 \times P_2^2$ (Brazil) x (France), Case 3: $P_1^1 \times P_2^3$ (Brazil) x (India), Case 4: $P_1^1 \times P_2^4$ (Brazil) x (Mozambique), Case 5: $P_1^1 \times P_2^5$ (Brazil) x (United State) and Case 6: $P_1^1 \times P_2^6$ (Brazil) x (South Sudan).

4.2 The effect of the control in the dynamics of fake news

In this subsection, we numerically explore the effects of the control proposed in the model (2.1) on the dynamics of the spreading of fake news.

Simulated Scenario B: In this scenario, we analyze the case where there is a control ($\xi_1 = \xi_2 = 1$) with an efficacy of $\rho_1 = \rho_1 = 0.1$ applied to both populations. The remaining parameters are the same as in simulated Scenario A.

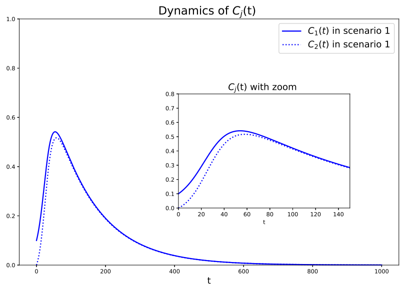
The dynamics of $C_j(t)$, for $j = 1,2$ in this scenario is presented in Figure 3. The numerical results in Figure 3 show that with a control with efficacy of $\rho_1 = \rho_2 = 0.1$ (10%) the number of people who believe in fake news circulating is drastically reduced (around 20%), compared to the dynamics of $C_j(t)$ without control (see Figure 1). Another phenomenon observed in the symmetric control scenario (**scenario B**) is that the dynamics of $C_j(t)$ near the peak remains monotone with respect to where fake news begins to spread, contrary to the observed phenomena in **Scenario A** (see Figure 1). Furthermore, the control reduces the time to share fake news, mainly among populations with a lower development index - compare Figures 3-1 (c)-(d). In particular, Figures 3-(d) show that if the fake news starts in population 2, then it does not spread to population 1 ($C_1(t) = 0$ in scenario 2).

Simulated Scenario C: Here, we simulate the situation where only one of the populations adopts the control in the model (2.1). The initial conditions and parameters are as defined in **Scenario A**. In all the simulated scenarios, the effective control is supposed to be constant and given by $\rho_1 = \rho_2 = 0.1$.

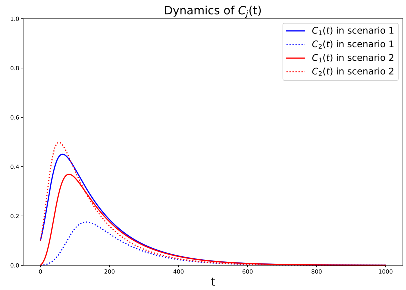
In Figure 4-(a)-(b) we present the dynamics of $C_j(t)$ for the populations of Brazil ($P_1 = P_1^1$) and the United States ($P_2 = P_2^5$). Figure 4 (a) only P_1 adopts the control. It can be seen in Figure 4 (a) that a delay is observed in the propagation of fake news in terms of where the fake news starts to be shared (if we are in scenario 1 or 2). Since Brazil's influence on the United States is not large, the dynamics of $C_2(t)$ is compared to the dynamics where there is no control at all in **Scenario A** (see Figure 1-(e)). The same conclusion can be obtained by observing the Figure 4 (b): the dynamics of $C_1(t)$ is almost the same as that without control presented in **Scenario A** (see Figure 1-(e)).

In Figure 5-(a)-(b) we present the dynamics of $C_j(t)$ for the populations of Brazil ($P_1 = P_1^1$) and Mozambique ($P_2 = P_2^4$). Figure 5 (a) only P_1 adopts the control. Since Brazil's influence on Mozambique is comparable to the influence that Brazil has on the United States (see Table 2), the conclusions about the dynamics of $C_2(t)$ in this case and the ones about Figure 4 (a) are the same for scenario 1. On the other hand, if fake news begins to spread in Mozambique and Brazil adopts the control (scenario 2 in Figure 5 (a)), then is is enough to isolate the country from fake news ($C_1(t) = 0$ in scenario 2). On the contrary, if Mozambique adopts the control, then a small impact can appear on the spread of false news in Brazil (compare Figure 5 (b) and Figure 1-(d)).

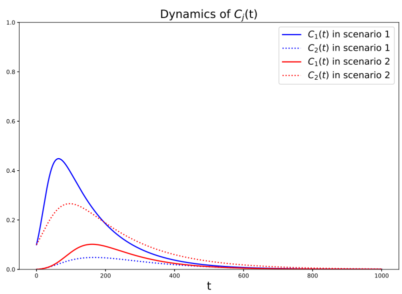
Observing the results in Figures 4-5, we conclude that for the populations with the highest development index that share information, adoption of the control in only one of them will not affect



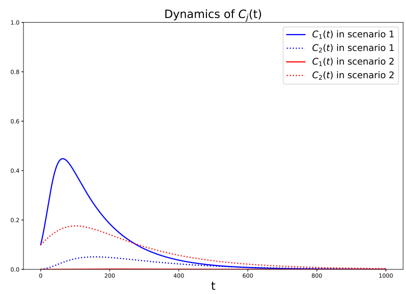
(a) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^1$ (Brazil) in **Scenario B**.



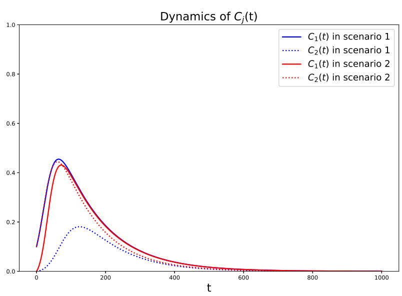
(b) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^2$ (France) in **Scenario B**.



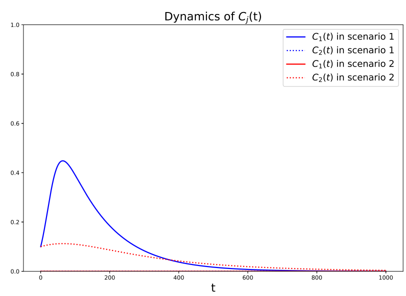
(c) Dynamics of C_j from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^3$ (India) in **Scenario B**.



(d) Dynamics of C_j, for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^4$ (Mozambique) in **Scenario B**.

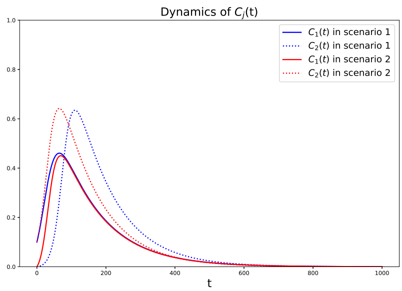


(e) Dynamics of C_j,for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^5$ (United State) in **Scenario B**.

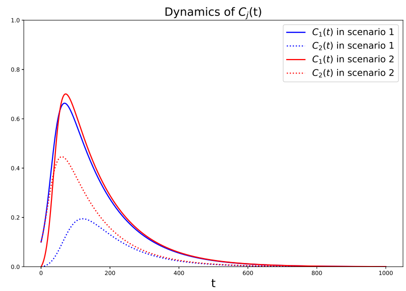


(f) Dynamics of C_j,for $j = 1,2$, from $P_1 := P_1^1$ (Brazil) and $P_2 = P_2^6$ (South Sudan) in **Scenario B**.

Figure 3: Dynamics of C_j, for $j = 1,2$, from **Scenario B** with parameters in Table 1 and 2, with symmetric control ($\xi_1 = \xi_2 = 1$) and symmetric efficacy ($\rho_1 = \rho_2 = 0.1$).



(a) Dynamics of C_j, for $j = 1,2$ from **Scenario C** with population $P_1 = P_1^1$ (Brazil) and $P_2 = P_2^5$ (United State), where only P_1 (Brazil) adopts the control ($\xi_1 = 1$ and $\xi_2 = 0$).



(b) Dynamics of C_j, for $j = 1,2$ from **Scenario C** with population $P_1 = P_1^1$ (Brazil) and $P_2 = P_2^5$ (United State), where only P_2 (United State) adopts the control ($\xi_1 = 0$ and $\xi_2 = 1$).

Figure 4: Dynamics of C_j, for $j = 1,2$ from the **Scenario C**, where $P_1 = P_1^1$ (Brazil) and $P_2 = P_2^5$ (United State).

the dynamics of the spread of fake news in the other. In this case, the recommendation is that both take some control measures, as presented in Figure 3. On the contrary, if populations have the highest contrast in the development index, then the more developed country can be isolated from fake news spread from the other population if it adopts some control strategy (5(a) scenario 2).

Simulated Scenario D: In the simulations presented below, we explore the effects on the dynamics of fake news when both populations adopt the control ($\xi_1 = \xi_2 = 1$) in the model (2.1) but the effectiveness is different ($\rho_1 \neq \rho_2$). The initial conditions and parameters are as defined in **Scenario A**. For simplicity, we restrict our analysis to the case where the populations are symmetric, with $P_1 = P_2 = P_1^1$ as presented in Figure 6.

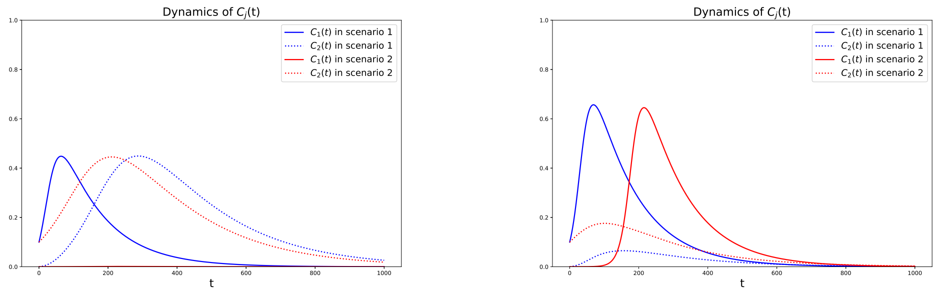
Figure 6 shows that the existence of control in both populations reduces the number of people who believe in fake news. However, it does not reduce the number of people who believe in fake news as in the case where the control is symmetric, as presented in Figure 3 (a).

4.3 Is the stiffness ratio a measure for the speed of the spread of fake news?

In [4, 6], the authors associate the speed of the spread and recovery of fake news with the ”time-dependent” inverse of the stiffness ratio of the Jacobian matrix of the proposed model. In this subsection, we show numerically that the association made by the authors in [4, 6] are not true in the case of two interacting populations.

Let the Jacobian matrix $JF(t,U(t))$ of model (2.1), as given in Proposition 3. The ”time-dependent” inverse of the stiffness ratio is given by

$$\tau(t,U(t)) := \frac{|\lambda_{min}(t,U(t))|}{|\lambda_{max}(t,U(t))|} \tag{4.1}$$



(a) Dynamics of C_j, for $j = 1,2$ from **Scenario C** with population $P_1 = P_1^1$ (Brazil) and $P_2 = P_2^4$ (Mozambique), where only P_1 (Brazil) adopts the control ($\xi_1 = 1$ and $\xi_2 = 0$).

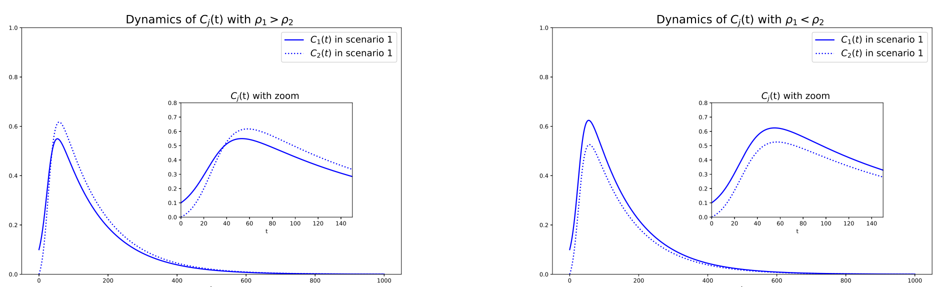
(b) Dynamics of C_j, for $j = 1,2$ from **Scenario C** with population $P_1 = P_1^1$ (Brazil) and $P_2 = P_2^4$ (Mozambique), where only P_2 (Mozambique) adopts the control ($\xi_1 = 0$ and $\xi_2 = 1$).

Figure 5: Dynamics of C_j, for $j = 1,2$ from the **Scenario C**, where $P_1 = P_1^1$ (Brazil) and $P_2 = P_2^4$ (Mozambique).

where $\lambda_{max}(t,U(t))$ and $\lambda_{min}(t,U(t))$ are, respectively, the maximum and the minimum non-zero eigenvalues of the Jacobian matrix $JF(t,U(t))$, for any $t \geq 0$.

In Figure 7, we show the corresponding evolution of $\tau(t,U(t))$ for the interacting populations and parameters as in **Scenario A**.

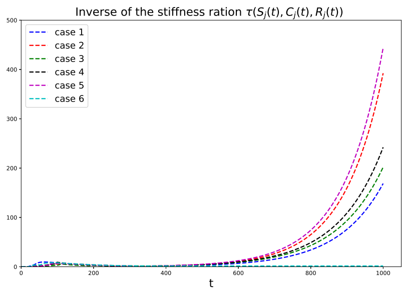
The results in Figure 7 suggest that, for populations that interact (differently from the case analyzed in [4, 6]), large values for $\tau(t,U(t))$ are obtained depending on where the disease started (see Cases 1 and 3 in scenarios 1 or 2). Moreover, the diseases spread slowlier in case 4 than in



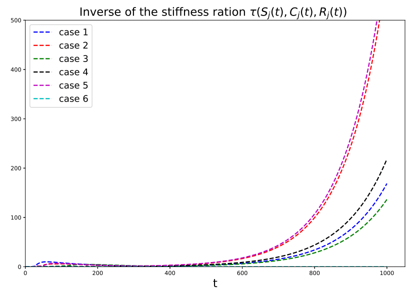
(a) Dynamics of C_j, for $j = 1,2$ from **Scenario D** with $\rho_1 = 0.1$ and $\rho_2 = \rho_1/2$.

(b) Dynamics of C_j, for $j = 1,2$ from **Scenario D**, with $\rho_2 = 0.1$ and $\rho_1 = \rho_2/2$.

Figure 6: Dynamics of C_j from the **Scenario D**, where $P_1 = P_2 = P_1^1$ (Brazil).



(a) $\tau(t,U(t))$ for scenario 1 in the simulated **Scenario A**.



(b) $\tau(t,U(t))$ for scenario 2 in the simulated **Scenario A**.

Figure 7: $\tau(t,U(t))$, with interacting populations for the simulated **Scenario A**. The cases presented are as follows: Case 1: $P_1 = P_2 = P_1^1$ (Brazil) x (Brazil), Case 2: $P_1^1 \times P_2^2$ (Brazil) x (France), Case 3: $P_1^1 \times P_2^3$ (Brazil) x (India), Case 4: $P_1^1 \times P_2^4$ (Brazil) x (Mozambique), Case 5: $P_1^1 \times P_2^5$ (Brazil) x (United State) and Case 6: $P_1^1 \times P_2^6$ (Brazil) x (South Sudan).

cases 1 or 3, but the values $\tau(t,U(t))$ are higher for case 4 than for cases 1 or 3. Therefore, the values of $\tau(t,U(t))$ as a measure of the velocity of diffusion of fake news are not valid for the case of two interacting populations (see also [18]).

5 CONCLUSIONS AND FUTURE DIRECTIONS

This paper presents a reinterpretation of a compartmental SIR-type model (where the parameters are related to the human development index and population influences) to analyze the spread of fake news among two distinct subpopulations that share information. We prove the well-posedness of the proposed model and derive the simplest equality - (3.2) -that shows the impact of the influence of a population (country in this approach) to another in the number of people that will believe in fake news. The results of the numerical simulations verify the theoretical findings. Furthermore, they suggest that the speed of diffusion of fake news is significantly affected by the gap between the human development indices of each population and the index of influence of one population on another. In particular, a small amount of control over the information shared by the population leads to a considerable decrease in the amount and velocity of fake news diffusion. Moreover, the inverse of the stiffness ratio is not monotonic among two inhomogeneous interacting subpopulations, which affects the speed of spreading of fake news, in contrast to the findings in [4, 6].

The findings of this paper will be supplemented in future studies with an examination of stability and bifurcation. Additionally, the concepts discussed here can be applied to populations that interact in a network [12, 14].

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