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Fuzzy Modelling to Describe the Pollutant Concentration in Fluids

M. S. ALCINO[1], S. A. B. SALGADO[2*], D. M. PIRES[3], S. M. DE SOUZA[4], R. A. ARMINDO[5] and N. C. DA SILVA[6]

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ABSTRACT. The study of the concentration dynamics of a pollutant substance in a fluid is a classic problem of fluid mechanics given by the transport equation $u_t + cu_x = 0$, where $u = u(x,t)$ denotes the pollutant concentration along a horizontal pipe of a fixed cross-section in the positive x direction at he time $t > 0$ and c represents the fluid propagation velocity. In view of that, the velocity of propagation of the fluid is a physical quantity, obtained, generally in an approximate form, which makes such quantity uncertain. In this paper, we propose to obtain the concentration when the constant c represents the fuzzy set. The concentration was obtained by using the Zadeh's Extension Principle. Through the concentration obtained, we analyze the influence of uncertainty on the fluid propagation velocity in the concentration dynamics and explore possible practial applications in case-studies of engineering, environmental and soil sciences.

Keywords: Zadeh's extension principle, transport equation, partial differential equations, fuzzy number.

1 INTRODUCTION

Partial differential equations represent a fundamental role in solving problems related to engineering, physics, biology, finance, among others. In particular, analyzing and understanding partial differential equations that arise in fluid dynamics is essential for many industrial applications such as underground reservoir oil recovery, heat exchangers and chemical reactors. A major

*Corresponding author: Silvio Bueno Antonio Salgado – E-mail: silvio.salgado@unifal-mg.edu.br

[1]Institute of Applied Social Sciences, Federal University of Alfenas, Celina Ottoni, 4000, 37048-395, Varginha, MG, Brazil – E-mail: osaraivamatheus@gmail.com https://orcid.org/0000-0003-4914-1202

[2]Institute of Applied Social Sciences, Federal University of Alfenas, Celina Ottoni, 4000, 37048-395, Varginha, MG, Brazil – E-mail: silvio.salgado@unifal-mg.edu.br https://orcid.org/0000-0002-1231-9946

[3]Institute of Applied Social Sciences, Federal University of Alfenas, Celina Ottoni, 4000, 37048-395, Varginha, MG, Brazil – E-mail: danilo.pires@unifal-mg.edu.br https://orcid.org/0000-0001-6821-3748

[4]Physics Department, Federal University of Lavras, Aquenta Sol, 37200-900, Lavras, MG, Brazil – E-mail: sergiomartinsde@ufla.br https://orcid.org/0000-0002-1922-4456

[5]Physics Department, Federal University of Lavras, Aquenta Sol, 37200-900, Lavras, MG, Brazil – E-mail: robson.armindo@ufla.br https://orcid.org/0000-0003-4675-8872

[6]Federal Center for Technological Education, Av. dos Imigrantes, 37022-560, Varginha, MG, Brazil – E-mail: nsilva@cefetmg.br https://orcid.org/0000-0002-0885-5252

problem in fluid mechanics is the determination of the concentration of a pollutant substance released in a fluid that moves at a constant velocity c. The fluid velocity of propagation is a physical quantity, obtained, generally in an approximate form, which makes such quantity uncertain. The motivation for treating the velocity c as uncertain has its roots in the physical characteristics of the problem. The method considering the initial dispersion velocity calculated in most cases may not be accurate or vary according to the physical properties of the fluids under study. Thus, it is reasonable to consider the velocity c as uncertain considering the difficulty of its measurement.

There are several approaches to treat the uncertainty in the mathematical literature. The Fuzzy Set Theory was introduced in 1965 by Loft Asker Zadeh, [20] in order to give mathematical treatment to certain subjective linguistic terms, such as "approximately", "around", etc. The term fuzzy differential equation (FDE) was introduced by [6]. When modeling a problem via FDEs, we may have initial conditions, coefficients and/or parameters represented by fuzzy sets. In [18] it was proposed the type of fuzzy harmonic oscillator with initial conditions given by fuzzy numbers. In [14] the authors study the solution to a diffusion problem using the Zadeh extension principle. In [5] the authors consider elementary the solutions of some fuzzy partial differential equations using two distinct techniques. In [1] the authors use the Homotopy Perturbation Method (HPM) to find approximate solutions of fuzzy partial differential equations. In [2] the authors study the homogeneous and non-homogeneous transport equation considering the notion of strongly generalized differentiability. In [13] the authors study fractional discrete–time diffusion equation with uncertainty via fuzzy sets. The use of fuzzy set theory in differential equations, allows to obtain a more robust dynamics of the studied system.

In this paper, we propose to obtain the concentration $u = u(x,t)$, where

$$\begin{cases} u_t + cu_x = 0 \text{ in } \mathbb{R} \times (0, +\infty) \\ u = f \text{ on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1.1}$$

and the coefficient c represents the velocity of the fluid is the fuzzy number. Moreover, we consider two fluids flowing through a surface in which the initial concentration is given by a Gaussian on some specific points. Thus, the differential equation described in (1.1) becomes a partial differential equation with the fuzzy coefficient. Therefore, we intend to investigate the effect of the uncertainty attributed to the velocity c in the model dynamics. We will use the Zadeh's Extension Principle to extend the deterministic solution and then a defuzzification method is used to understand the effect of uncertainty on the c velocity on system dynamics.

The present paper is organized as follows. In the first part (Section 2), we present some fundamental concepts of fuzzy sets. In Section 3 we briefly discuss the classic transport equation and the choice of suitable parameters for the initial condition of the pollutant substance concentration. In Section 4 we study the homogeneous transport equation with fuzzy velocity via Zadeh's Extension Principle and through the defuzification method called center of gravity.

Finally, we present our conclusion in Section 5.

2 PRELIMINARY

In this section, we will present some fundamental concepts for the construction of this paper.

Definition 2.1. [3] *Let U be a topological space. A fuzzy subset A of U is characterized by a membership function $\mu_A : U \to [0,1]$, where $\mu_A(x)$ denotes the degree in which the element x belongs to the fuzzy set A.*

Note that a classical subset A of U, can be associated with a fuzzy subset whose membership function is given by the characteristic function $\chi_A(x)$. For notation convenience, we may use the symbol $A(x)$ instead of $\mu_A(x)$.

Definition 2.2.

[3] *The α-levels of the fuzzy subset A are defined by*

$$[A]^\alpha = \begin{cases} \{x \in U; A(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ \overline{\{x \in U; A(x) > 0\}} & \text{if } \alpha = 0, \end{cases} \tag{2.1}$$

where $\overline{X}$ denotes the closure of the subset X of the U.

Definition 2.3. [3] *A fuzzy subset A of $\mathbb{R}$ is a fuzzy number if it satisfies the following properties:*

i) all α-levels of A are closed and nonempty intervals of $\mathbb{R}$.

ii) the set $\{x; A(x) > 0\}$ is a bounded set of $\mathbb{R}$.

From Definition 2.3, the α-level of a fuzzy number A is represented by its interval endpoints

$$[A]^\alpha = \left[a_-^\alpha, a_+^\alpha\right], \tag{2.2}$$

for all $\alpha \in [0,1]$.

We use the symbol $\mathscr{F}(U)$ to denote the fuzzy subsets of U and $\mathbb{R}_\mathscr{F}$ to denote the set of all fuzzy numbers. As an example of a fuzzy number we have the triangular fuzzy number, whose α-levels are given by $[A]^\alpha = [(m - a_-^0)\alpha + a_-^0, (m - a_+^0)\alpha + a_+^0]$, for all $\alpha \in [0,1]$, where $[A]^0 = [a_-^0, a_+^0]$ and $\{m\} = [A]^1$ (a real number). A triangular fuzzy number is denoted by the triple $(a_-^0; m; a_+^0)$.

Definition 2.4. [8] *We say that a fuzzy set A of $\mathbb{R}$ is symmetrical about $x \in \mathbb{R}$ if $A(x-y) = A(x+y)$, for all $y \in \mathbb{R}$. If there is no $x \in \mathbb{R}$ that satisfies this property, we say that A is non-symmetric.*

The Zadeh's Extension Principle for a function $f : X \longrightarrow Z$ indicates how the image of a fuzzy subset A of X is computed when the function f is applied. It is expected that the image will be a fuzzy subset of Z.

Definition 2.5. [3] *Let f be a function such that $f : X \longrightarrow Z$ and let A be a fuzzy subset of X. Zadeh's extension of f is the function $\hat{f}$ which applied to A gives us the fuzzy subset $\hat{f}(A)$ of Z with the membership function given by*

$$\widehat{f}(A)(z) = \begin{cases} \sup\limits_{x \in f^{-1}(z)} A(x), & \text{if} \quad f^{-1}(z) \neq \emptyset \\ 0, & \text{if} \quad f^{-1}(z) = \emptyset, \end{cases} \tag{2.3}$$

where $f^{-1}(z) = \{x;\ f(x) = z\}$ is the preimage of z.

Theorem 2.1 indicates that the α-levels of the fuzzy set obtained by the Zadeh's Extension Principle coincides with the images of the α-levels by the crisp function.

Theorem 2.1. [15], [17] *Let $f : X \longrightarrow Z$ be a continuous function and let A be a fuzzy subset of X with non empty and compact α-levels. Then, the following equality holds*

$$[\hat{f}(A)]^\alpha = f\left([A]^\alpha\right), \tag{2.4}$$

for all $\alpha \in [0,1]$.

Let us introduce the Extension Principle for functions of two variables.

Definition 2.6. [3] *Let $f : X \times Y \longrightarrow Z$ be a function and let A and B be fuzzy subsets of X and Y, respectively. The extension $\hat{f}$ of f, applied to A and B is the fuzzy subset $\hat{f}(A,B)$ of Z with the membership function given by:*

$$\widehat{f}(A,B)(z) = \begin{cases} \sup\limits_{(x,y) \in f^{-1}(z)} \min\{A(x), B(y)\}, & \text{if} \quad f^{-1}(z) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(z) = \emptyset, \end{cases} \tag{2.5}$$

where $f^{-1}(z) = \{(x,y); f(x,y) = z\}$.

There are many defuzzification methods. Among them, the most used are: Center of Gravity (COG), Center of Area (COA), Mean of Maxima (MOM) and Lower of the Highs (LOH).

Definition 2.7. [4] *The center of gravity (COG) of a fuzzy subset $A \in \mathscr{F}(U)$ is a real number*

$$COG(A) = \frac{\int_W xA(x)dx}{\int_W A(x)dx}, \tag{2.6}$$

where $W = \{x;\ A(x) > 0\}$.

3 TRANSPORT EQUATION – CLASSIC MODEL

The transport equation is a first order partial differential equation that models the dynamics of the concentration of a pollutant in a fluid in a given physical space. In this study, physical space can be understood as a pipe. In addition, some physical characteristics of the problem will not be considered, such as fluid viscosity and atmospheric pressure inside the pipe. Consider a fluid, water, say, with constant propagation velocity c along a horizontal pipe of a fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Let $u(x,t)$ be its pollutant concentration at the time t. Then

$$u_t + cu_x = 0. \tag{3.1}$$

This means that the substance is transported to the right at a fixed velocity c. Each individual particle moves to the right at the velocity c. Taking $u(x,0) = f(x)$, we have:

$$\begin{cases} u_t + cu_x = 0 \text{ in } \mathbb{R} \times (0,+\infty) \\ u = f \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{3.2}$$

Theorem 3.2. [19] *The Problem*

$$\begin{cases} u_t + cu_x = 0 \textit{ in } \mathbb{R} \times (0,+\infty) \\ u = f \textit{ on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{3.3}$$

where $f \in C^1(\mathbb{R})$, admits solution as

$$u(x,t) = f(x - ct). \tag{3.4}$$

Theorem 3.2 allows us to obtain the concentration $u(x,t)$ from any initial concentration $f(x)$. Our objective is to study the concentration of a pollutant as a function of the time and in order to do that, we will consider two fluids flowing through a surface. The initial concentration, natural, for this situation is a Gaussian centered on specific points [12], given by:

$$f(x) = \frac{1}{2\sqrt{d\pi}} e^{-\frac{1}{4d}(x-x_0)^2}, \tag{3.5}$$

where $(x, x_0) \in \mathbb{R}^2$ and d is a positive number, measured in m^2. Thus, through $f(x)$, by Theorem 3.2 it is possible to obtain the concentration $u = u(x,t) = f(x - ct)$. It is important to note that we are working with a unit concentration, that is, $\int_{-\infty}^{+\infty} f(x)dx = 1$.

Figure 1 is the graphical representation of the pollutant concentration $u = u(x,t)$, using (3.5) as the initial condition and $d = 1$, $x_0 = 0$. Without loss of generality, we consider $d = 1$, since $\sqrt{d}$ has a unit of length, we can always change the time unit by one of its multiples (or submultiple), arbitrarily (as in the case of the length unit).

Through the Figure 1 we observed a maximum pollutant concentration in the initial positions, as well as an inverse relationship between the pollutant concentration and the time variable. This movement is explained by the d parameter of the initial condition.

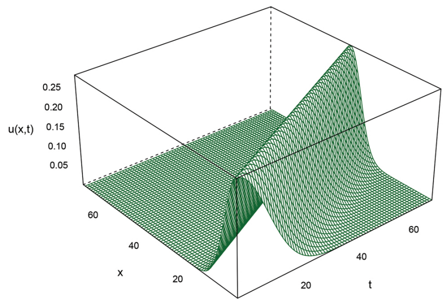

Figure 1: Graphical representation of the pollutant concentration $u = u(x,t)$ of the Problem (3.3) where $f(x)$ is given by (3.5) with $d = 1$ and $x_0 = 0$.

4 RESULTS AND DISCUSSION

In this section, we study the homogeneous transport equation with fuzzy velocity, *i.e.,*

$$\begin{cases} u_t + cu_x = 0 \\ u = f, \end{cases} \tag{4.1}$$

where $c \in \mathbb{R}_{\mathscr{F}}$ and f is given by (3.5) where $d = 1$ and $x_0 = 0$.

The motivation for treating the velocity c as uncertain has its roots in the physical characteristics of the problem. The method as the initial dispersion velocity which is calculated in most cases may not be accurate or vary according to the physical properties of the fluids under study. Thus, it is reasonable to consider the velocity c as uncertain regarding the difficulty of its measurement.

Besides the fuzzy theory, there are several methods to treat the uncertainty as statistical modeling. However, frequentist or Bayesian statistical approaches require information that is hard to access. For example, when treating the velocity c as a random variable in a stochastic continuous indexing process, it would be necessary to find a probability distribution to it or to observe a large number of experiments in a controlled environment to describe its behavior. Therefore, in this paper, we propose the mathematical treatment of the velocity c through the fuzzy set theory, which deals with uncertainties that are not derived from the observation of a large number of events (repetition). The following result establishes the α-levels of the concentration $u(x,t)$ given by (4.1).

As the consequence of Nguyen's Theorem [15], we have:

Theorem 4.3. *The α-levels of the concentration $u(x,t)$ solution of the (4.1) is given by*

$$[u(x,t)]^{\alpha} = [\min f(s), \max f(s)], \tag{4.2}$$

where $s \in [x - c_{-}^{\alpha}t, x + c_{+}^{\alpha}t]$ and $[c]^{\alpha} = [c_{-}^{\alpha}, c_{+}^{\alpha}]$, for all $\alpha \in [0,1]$.

Proof. According to Theorem 3.2 we fuzzified the real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ to the function $\hat{f} : \mathbb{R}_{\mathscr{F}} \longrightarrow \mathbb{R}_{\mathscr{F}}$ through the Zadeh's Extension Principle. Since f is a continuous function, we have $[\hat{f}(A)]^{\alpha} = f([A]^{\alpha})$. For instance, let $[A]^{\alpha} = [x - c_{-}^{\alpha}t, x + c_{+}^{\alpha}t]$ and f be a nondecreasing function. Then we have

$$[u(x,t)]^{\alpha} = [\hat{f}(A)]^{\alpha} = f([A]^{\alpha}) = [\min f(s), \max f(s)],$$

where $s \in [x - c_{-}^{\alpha}t, x + c_{+}^{\alpha}t]$, for all $\alpha \in [0,1]$. □

Remark 1. *In this work, we use the Zadeh's Extension Principle to find the solution of the transport equation with the fuzzy coefficient. It is possible to compare the solution found via the Zadeh's Extension Principle with the solution obtained using the Hukuhara derivative or even with differential inclusions, for example. Differential inclusions and the Zadeh Extension method under certain conditions produce the same solutions [3]. On the other hand, according to [2] if* $f \in C^2(\mathbb{R})$ *is an integrable nonnegative monotone function and* $\hat{f} : \mathbb{R}_{\mathscr{F}} \longrightarrow \mathbb{R}_{\mathscr{F}}$ *be the Zadeh's extension of* f *and* $c \in \mathbb{R}_{\mathscr{F}}$ *with* $\{x \in \mathbb{R};\ c(x) > 0\} \subset (0, +\infty)$, *$f$ and f' nondecreasing functions, then u is Hukuhara differentiable it is solution of the Problem (4.1).*

Figure 2 illustrates the graphical representation of the fuzzy concentration $[u(x,t)]^{\alpha}$ as a function of the time, with the propagation velocity fuzzy c, where $c = (1.8; 2; 2.2)$. In view that we are not interested in studying a specific fluid, without loss of generality, we studied the system in units of d, that is, $d = 1$. In this paper we are not interested in studying specific pollutants and fluids, so we chose to treat the pollutant propagation velocity as a number around two. If the fluid in question was specified, physical properties such as viscosity, pressure, etc. should be considered to make the best choice of the fuzzy number c. Note the uncertainty is present in the coefficient c and not in the amount of pollutant that is fixed, that is, the maximum concentration is limited to a certain value.

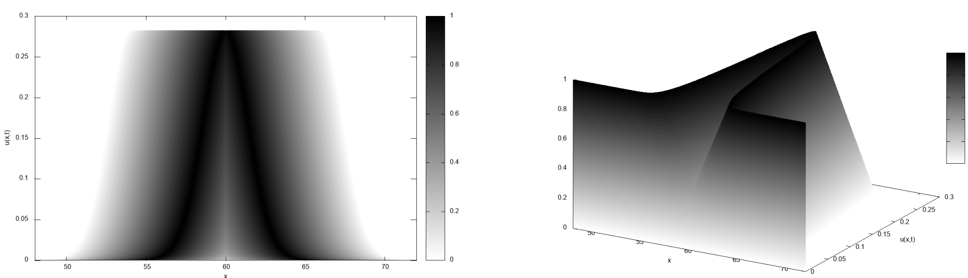

Figure 2: Graphical representation of the fuzzy pollutant concentration presented in (4.2), where $f(x)$ is given by (3.5) with $d = 1$, $x_0 = 0$ and $t = 30$ for $c = (1.8; 2; 2.2)$. The α-levels are represented by the gray scale ranging from 0 to 1 and are represented respectively by a color variation from white to black.

According to the shape of the triangular fuzzy number used to incorporate uncertainty into the velocity c, it is noted by the scale next to the graph illustrated by Figure 2 that for each position

x of the surface at a fixed time, there is a concentration $u = u(x,t)$ of pollutant with membership $\alpha \in [0,1]$.

The geometric shape of the fuzzy number c is crucial for understanding the diffusion phenomenon studied in this paper. Non-symmetrical fuzzy triangular number c may establish different rates of pollutant dispersion at different times, which in turn can be explained by observing physical fluid characteristics such as viscosity, surface friction, etc. In Figure 3 we have the graphical representation of the concentration by analyzing non-symmetric triangular fuzzy numbers c. That is, the triples $(1.8;2;2)$ and $(2;2;2.20)$ respectively. The darkest region of the graph represents the α-level of the concentration $u(x,t)$.

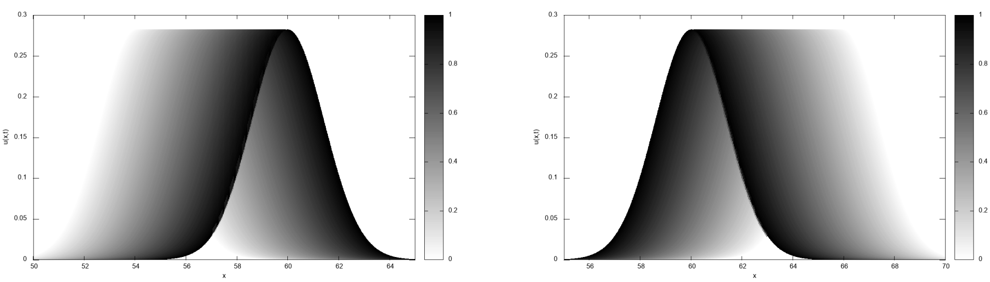

Figure 3: Graphical representation of the fuzzy pollutant concentration presented in (4.2) in the xu plane where $f(x)$ is given by (3.5) for $d = 1$, $x_0 = 0$, $t = 30$, $c = (1.8;2;2)$ and $c = (2;2;2.20)$ respectively. The α-levels are represented by the gray scale ranging from 0 to 1 and are represented respectively by a color variation from white to black.

The triangular fuzzy numbers chosen could have other shapes, having bases with larger or smaller amplitudes. The implications of this would be $u(x,t)$ curves with more or less uncertainty. So the curves will be close (or not) to the deterministic case as we show in Figure 1 in any time and position.

In Figure 4 the graphical representation of the defuzzified concentration given by (4.1) is shown for each triple tested. The center of gravity (COG) was used as a defuzzification method. The classic concentration for comparison, in this case, was obtained from (3.5) with $d = 1$, $x_0 = 0$, $t = 30$ and $c = 2$.

It is noticed that the defuzzified concentrations have points of intersection with the classic concentration, but there is no point where all concentrations overlap. In addition, all defuzzified concentration has a lower maximum concentration value than in the classic case, which represents an indication that the uncertainty considered in the fluid velocity influences the intensity of the concentration along the x axis. On the other hand, by decreasing the amplitude of the uncertainty in the velocity c, the defuzzified concentrations approximate the values of the classical concentration, since the classical concentration is considered a fuzzy model with an amplitude equal to 0.

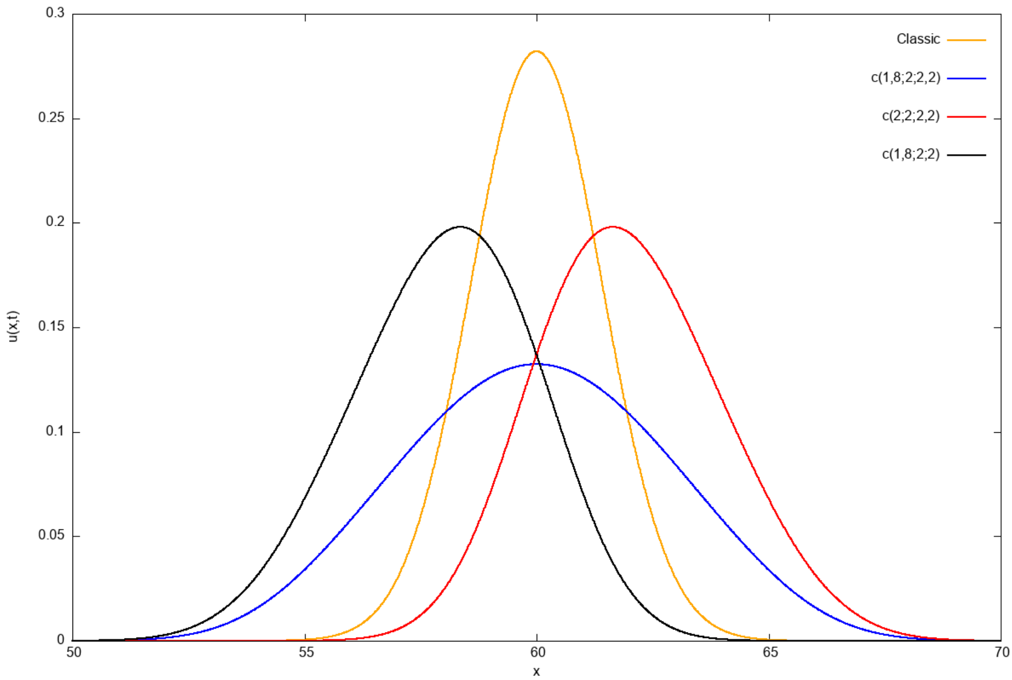

Figure 4: Graphical representation of the classic concentration $u(x,t)$ and the defuzzified concentration for $d = 1$, $x_0 = 0$, $t = 30$, $c = (1.8;2;2.2)$, $c = (2;2;2.2)$ and $c = (1.8;2;2)$.

In Figure 5 we show the defuzzified concentration and the classic concentration at several different times, with the aim of showing the temporal evolution of the effects generated by the fuzzy velocity. In standard situations, uncertainties in velocity are usually small, so in most cases we will use a symmetrical fuzzy velocity given by $c = (1.9;2;2.1)$, the classic velocity is $c = 2$ and the total amount of pollutant is unitary[1]. We do not show the instant $t = 0$, because in this case the fuzzy concentration is equal to the classic concentration. At $t = 10$, Figure 5(a), we can see that the defuzzified concentration is already visibly different from the classic concentration, mainly because the peak of the defuzzified concentration is less than the peak of the classic contraction. At $t = 50$, Figure 5(b) the defuzzified concentration is already quite different from the classic concentration, reinforcing a tendency to decrease the peak value and a considerable increase in the base of the defuzzified concentration. At $t = 100$, Figure 5(c) the defuzzified concentration continues to maintain the trend of decreasing the maximum peak value, but with the emergence of two symmetrical points of maximum concentration. This is an indication that associated with the fuzzy velocity, there is a process dividing the concentration into parts. At $t = 400$, Figure 5(d), the defuzzified concentration confirms the decrease in the maximum peak value, reinforces the tendency to divide the concentration into parts, only now with a third region where the defuzzified concentration is practically constant. At $t = 800$, Figure 5(e), the decrease in the maximum

[1] $\int_{-\infty}^{\infty} u(x,t)_{cla} dx = \int_{-\infty}^{\infty} u(x,t)_{def} dx = 1$

value of the peak of the defuzed concentration is confirmed, a homogenization trend is shown, and the concentration is divided into 3 parts with their well-defined maxims, reinforcing that associated with the fuzzy velocity, there is a process of division of concentration. However, there are physical situations in which the uncertainty in velocity can be considerable, such as the velocity of the pollutant that moves at the same speed as the flow of fluid by convection. In these cases, the average flow velocity V_m is given by $V_m = \frac{1}{S}\int_S \langle \vec{V}, \vec{n} \rangle dS$, where S is the cross-sectional area formed by the flow current pipe and $\langle \vec{V}, \vec{n} \rangle$ is the scalar product between the velocity and the normal unit vectors. This strategy is used due to the uncertainty about the velocity, which varies with the position along the S section of the flow in small sections such as those of building or water mains greater in large sections of flow, such as tsunamis, via wave propagation towards the coast. With the amplitude of the wave and the flow velocity, the destructive force of the tsunami is inferred, which is consequently uncertainty, since the flow velocity along the great wave is uncertain. This uncertainty occurs due to the several variations existing along the wave amplitude, such as: position along the flow, concentration of the temperature, viscosity, turbulence and vorticity. Another way to observe the effects generated by a high uncertainty is to consider a small uncertainty in velocity and a long period, For these reasons and to highlight the effects related to the fuzzy velocity, we considerer $t = 10000$ and $c = (1.9; 2; 2.1)$, Figure 5(f), where all the trends observed in the previous figures. The first factor to be highlighted is the fact that we have two symmetrical phases with the same properties one early and another delayed in relation to the classic concentration. Analyzing the advanced phase, a first region with many pick values of defuzzified concentration decreasing until reaching a second region. In the second region, we have again a large number of peak values of defuzzified concentration, but decreasing in a smooth way, until reaching a third region. In the third region, we have a large number of peak values of defuzzified concentration decreasing, but now to zero. This graph confirms that, associated with fuzzy velocity, there is a process of dividing the fuzzy-defuzzified concentration into parts, showing a physics much richer than that associated with the traditional classical process.

In Figure 6 we show the defuzzified concentration and the classic concentration at several different times, with the aim of showing the temporal evolution of the effects generated by the fuzzy velocity. In standard situations, uncertainties in velocity are usually small, so in most cases we will use an non-symmetric fuzzy velocity given by $c = (2; 2; 2.2)$, the classic velocity is $c = 2$. At $t = 10$, Figure 6(a), we can see that the defuzzified concentration is already visibly different from the classic concentration, mainly because the defuzzified concentration is a little lower and a little ahead of the peak of classic contraction. The advance is related to the non-symmetric of the fuzzy velocity. At $t = 50$, Figure 6(b), we can see that the defuzzified concentration is already quite different from the classic concentration, reinforcing the trends of advance and decrease of the peak in relation to the classic contraction peak. Another important trend to be observed is the increase in the base of the defuzed concentration, which is also related to the fuzzy velocity. At $t = 100$, Figure 6(c), we can observe the same characteristics as in Figure 6(b), plus a clear non-symmetric of the defuzzified concentration. It is important to emphasize that the non-symmetric in the concentration is an indicative of separation in parts of the defuzzified concentration. At $t = 400$, Figure 6(d), the defuzzified concentration confirms the tendency to decrease the maxi-

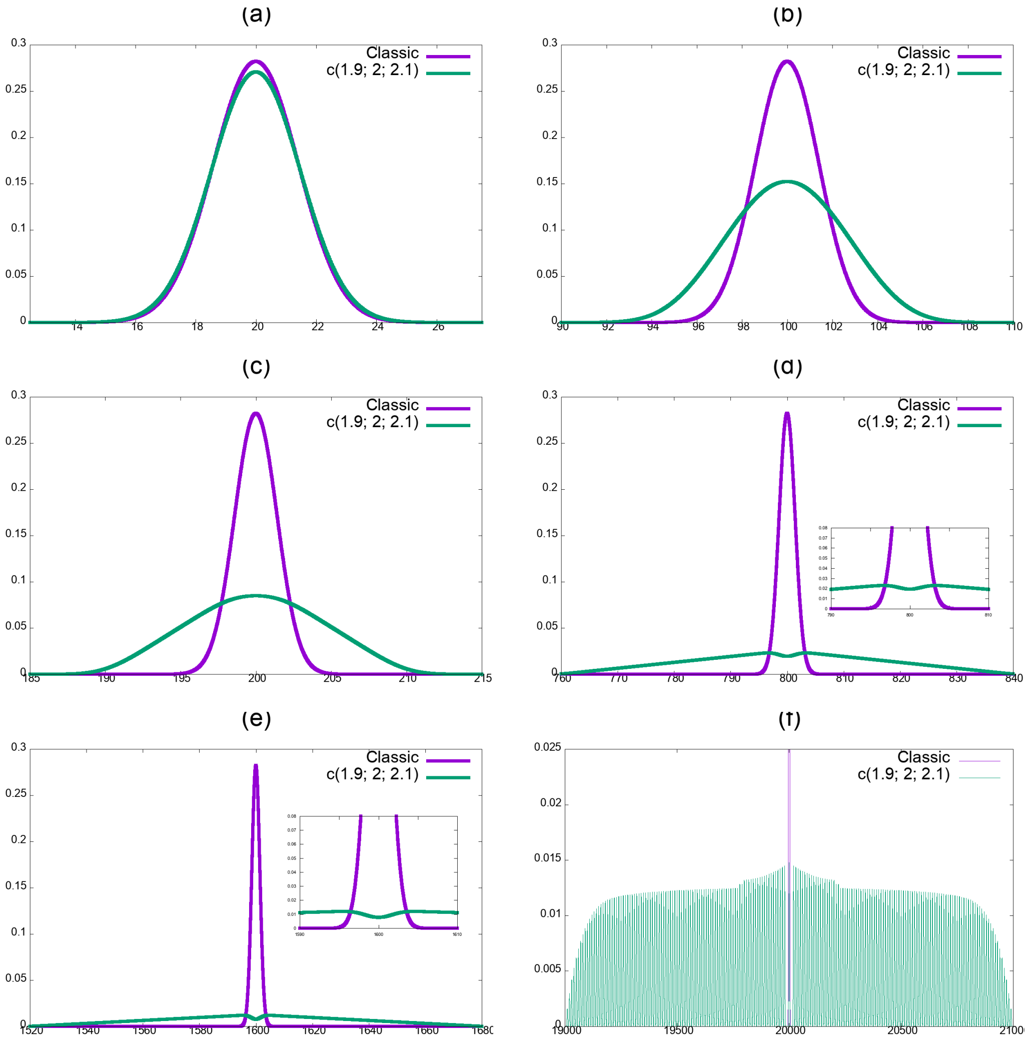

Figure 5: Defuzzified concentration for symmetrical fuzzy velocity $c = (1.9;2;2.1)$ and the classic concentration with velocity $c = 2$, $d = 1$ and a) $t = 10$, b) $t = 50$, c) $t = 100$, d) $t = 400$, e) $t = 800$. In f) to highlight the effects of fuzzy velocity on the process, we consider $c = (1.9;2;2.1)$ and $t = 10000$.

mum peak value, but with the appearance of a new peak, showing the separation in two parts of the defuzzified concentration. This is an indication that the fuzzy velocity may be associated with a separation process in parts of the concentration. For, $t = 800$ (graph e), the trends observed in the previous graphs are reinforced, but now with the concentration much more spatially spread, indicating a trend towards homogenization. However, we can clearly see that there is a region between the two parts mentioned above, in which the concentration is zero.

For $t = 10000$, Figure 6(f), we can clearly see, and with greater intensity, all the trends observed in the previous figures. The first factor to be highlighted is the fact that we have an advanced phase in relation to the classic concentration. Analyzing the advanced phase, we can observe a first region with many values of defuzzified concentration peaks decreasing rapidly until reaching a second region. In the second region, we have one more time a large number of peak values of defuzzified concentration, but decreasing smoothly, until reaching a third region. In the third region, we have a large number of peak values of defuzzified concentration decreasing rapidly, but now up to zero. As observed for symmetric fuzzy velocity, Figure 5(f), this graph confirms that, associated with fuzzy velocity, there is a process of dividing the fuzzy-defuzzified concentration into parts, showing a much richer physics than that associated with the traditional classical process.

4.1 Motivation, novelty, and possible practical applications

As motivation to develop this study, we list possible scenarios where our developed formulations may be applied highlighting study-cases in several major sciences such as engineering, soil and environmental sciences, and hydrology, where the transport of pollutant particles in the flow of fluids occurs.

Open channel flow

Traditionally, the described formulation of (3.4) is valid for the convective flux assigning for $f(x)$ the Gaussian distribution presented in (3.5), which is applied whether the velocity c is constant to the pollutant and the flow of a channel (or river) describes the concentration $u(x,t)$ without scattering behavior (diffusion). Thereby, the Gaussian distribution is propagated without changing its width or height.

In practical circumstances, this behavior will occur if the water velocity in the channel is constant and uniform for a small period of time and space becoming feasible the analysis of pollutants that are present in effluents. However, for a large variation or high turbulence of the fluid the found formulation in (3.4) has a considerable limitation and may not be applied because the diffusion should not be neglected. Furthermore, if c is constant, a mere analytical solution is found and therefore numerical solutions are not necessary. In (3.5), d increases with time but in this case it is not part of the solution since it is fixed and kept the same in the initial condition. The value of $d = 0$ denotes an infinite concentration at a single point of the fluid flow and when $d \longrightarrow \infty$ a uniform distributed concentration of the pollutant is found. The knowledge that d is about the squared channel width is another point of high relevance.

For rivers or lakes, the analysis should be done based on the ratio of diffusion and convective flows. However, the diffusion in lakes is quite small but the velocities are even smaller, about the order of cm/h. The differential equation that describes the convective/diffusive transport of pollutants in transient and non-transient flow regimes, with numerical solutions, is already known. Alternatively, the formulation presented in this work does not consider, in a first impression, the

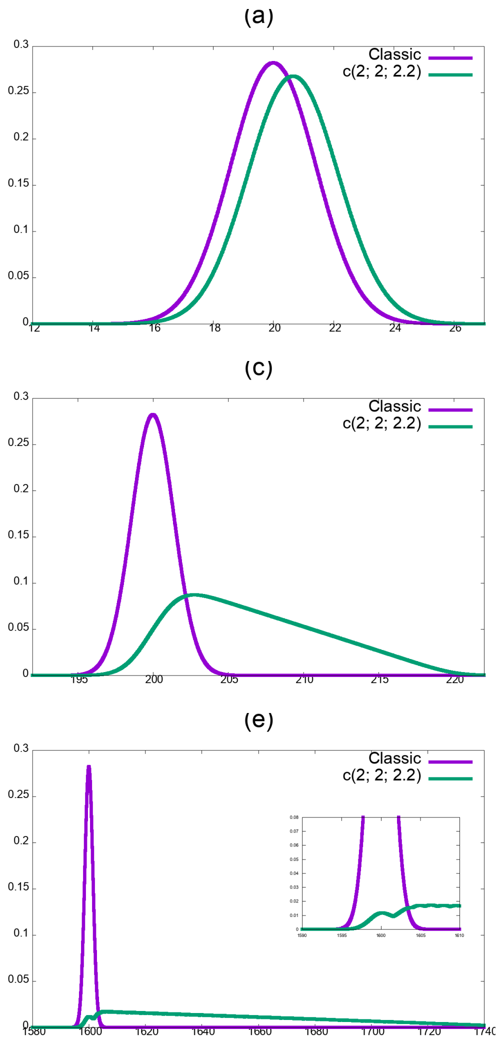

Figure 6: Defuzzified concentration for non-symmetric fuzzy velocity $c = (2;2;2.2)$ and the classic concentration with velocity $c = 2$, $d = 1$ and a) $t = 10$, b) $t = 50$, c) $t = 100$, d) $t = 400$, e) $t = 800$. In f) to highlight the effects of fuzzy velocity on the process, we consider $c = (2;2;2.20)$ and $t = 10000$.

piston flow effect but it does consider the velocity uncertainty relating it with the fuzzy set theory. Thus, the scattering effect (diffusion) in the pollutant dynamic is somehow considered and the velocity uncertainty has direct impact on the final equation solution, resulting in a more spread concentration cloud of c data. Someway, this velocity variation can compensate the diffusion, since it has a problem of mass conservation in the classic diffusion phenomenon, compensating the horizontal spread with the flattening of the initial Gaussian distribution provided in (3.5).

This is a classic issue with complete solutions applied in studies of hydraulics and hydrology (channels) and of atmospheric pollutants.

Soil contamination

Analyzing soil contamination, when soil columns with solutes are assessed in the laboratory, the infiltration rate or Darcy's velocity [7, 10] is applied because of the uncertainty in the convective flow of the solution water+pollutant. The capillary and larger soil pores are not perfect pipes, ranging their geometry in space and time and, therefore, result in velocity uncertainty. However, when the breakthrough curve solution is taken into account, the diffusion effect of the solute also be considered, mainly the horizontal one due to the capillary effects. Inert solutes do not react with the soil structure; thus, the convective is the significant effect in this transport, since the diffusion should only be considered when the concentration gradients highly differ. Thus, our formulation may also be applied to practical unidimensional circunstances of fluids transport, in which the wet front with effluent concentration moves towards in the x direction (for all y and z constants). This case supposes that the transport is high and given by the Darcy's law [7, 10] when the pressure gradient is known and the capillary diffusion neglected in the other directions. It is also possible to applied it in two-dimensional problems (x,z), where the concentration varies only in the x direction and the pollutant quickly reaches the water table, observing its displacement in the x direction by pressure gradients of Darcy. This approach neglects the concentration variations by diffusion in the z direction and it is interesting when the water table is close to the soil surface (shallow with rock at the bottom). In this example, the pollutant should has already reached the bottom of the soil and show a uniform distribution over z, depending on the Darcy's flow in its essence.

Besides the traditional convective and diffusive effects considered in the breakthrough curves in soil pollutant studies, there are also the interaction mechanisms of the solute with the soil (adsorption or desorption) and the chemical and biological transformations of these solutes that are often neglected. The formulation with constant velocity c, which only considers the existence of the convective effect (3.4), again implies a short space and period of time (x,t) or in breakthrough curves with pulses as piston flow effect. However, the treatment of (3.4) by the fuzzy set theory associates that c is defined in degrees of pertinence due to the uncertainty, which contributes again with the diffusive effect.

Future studies should analyze the breakthrough curves fitted with measured data taking use of the formulation presented in this study and compare them with the traditional formulations fitted with the same data including both convective and diffusive, since Darcy's velocity is uncertain due to the irregularity of the soil porous system. Uncertainty of c within the traditional physical mathematical models consider c as an apparent mean velocity.

Novelty

The formulation presented in this paper is based on an algebra associated with the theory of fuzzy sets, being away from the traditional physical-mathematical treatment utilized in applied physics for engineering, environmental, and agricultural sciences. With this in mind, it is now possible to assume the fuzzy velocity value as an interactive result of the pollutant with the soil. Thus, this interpretation can be useful, without the need of establishing dispersive/diffusive coefficients. The traditional formulation for breakthrough curves presented by several authors such as [9, 11, 16] shows a behavior similar to the first half of the Gaussian distribution format (3.5). If the difference between these formulations is statistically negligible, our solution by the fuzzy set theory can be also used to subtract the total effect (experimentally measured) by the single convective effect to better examine the dispersion/diffusion behavior of a given pollutant, which is of complex analysis in soils of several watersheds. Thus, this work might allow the inverse mapping of convective/dispersive effects to be successful in cases where these effects are of complex determination.

5 CONCLUSIONS

We studied the model known as the transport equation by introducing uncertainty in the coefficient related to the fluid velocity, denoted by c. To do so, we applied the fuzzy set theory to address the uncertainty proposed in the problem and suggested a solution based on the Zadeh Extension Principle. The uncertainty of the developed coefficient c was inserted by fuzzy triangular numbers, taking one as symmetrical and the others as non-symmetrical. Each of these results in a distinct uncertainty setting confirmed through disabled solutions and the difference between each solution compared to the classic case. As future work, we intend to study the non-homogeneous version of the transport equation using interactive fuzzy number arithmetic and the fuzzy Laplace transform. Futhermore, possible practical applications in soil pollutant transport studies, hydrology, engineering, and environmental sciences using the presented solution were discussed and should be further investigated. Thus, we hope that in the near future, new alternatives for solving traditional solutions, e.g., the formal equation formulated in breakthrough curves for situations of interest. This result opens alternatives for classical numerical solutions, where the mean velocity is addressed due to the velocity uncertainty in real phenomena. In this case, the ideal is to use the arithmetic of interactive fuzzy numbers, since the transport equation $u_t + cu_x = 0$ in the fuzzy case is a partial differential equation that contains interactivity. Under certain conditions, the diameter of the numerical solution decreases, thus obtaining more precise approximations of the temporal evolution of the studied dynamic system.

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