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On the Separate Assessment of Structural Effects on the Simple Beam Deflection in the Light of Fractional Calculus

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ABSTRACT. Euler-Bernoulli (EB) and Timoshenko-Ehrenfest (TE) theories model simple beams under linear constraints. But even keeping these constraints, specific structural effects in real applications compromise the accuracy of the models, such as stress concentration due to force reactions on the support contacts or bucking, for example. Both EB and TE solutions assume planar cross sections and structural stability, and therefore do not address those particular effects; the interest in using them is to explore the conditions under which shear effects are significant or not. Numerical solutions such as the ones obtained from the Finite Element Method (FEM) reach structural effects quite well, depending on the complexity of the problem and degree of refinement. However, although accurate, the numerical solutions do not distinguish whether a particular effect is on charge or not; they implicitly encompass all of them as a whole. To diagnose them for simple beams, analytical solutions such as EB or TE can be employed as long as they noticeably differ from the corresponding affected structures modeled in a FEM environment, the latter taken as reference. To measure this disagreement one can either directly compare each resulting deflection profile or, alternatively, compare the analytical solutions EB or TE against the corresponding fractional calculus ones fed with the FEM data. The second case, focus of this study, provides a better measurement, to our understanding, as the fractional order informs the magnitude of the effect not depending on absolute values, which allows a fairer evaluation. Each structure effect must be isolated so that to obtain the individually consequent order deviation from the original integer value. A separate assessment using fractional calculus is proposed in this work to, first, evaluate the effects of stress concentration on the support contacts, and, second, to prepare the modeling to potentially reveal another structural effect. Consistent results have been achieved.

Keywords: fractional calculus Euler-Bernoulli, Timoshenko, Timoshenko-Ehrenfest, ANSYS.

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1 INTRODUCTION

Simply supported beams can be accurately modeled by the Euler-Bernoulli (EB) theory if linear conditions are ensured and shear effects are negligible. The Timoshenko-Ehrenfest (TE) theory, nevertheless, stands out covering the shear effects in beams with relevant vertical dimension, as it takes into account the cross section rotation with respect to the neutral plane. The interest in this work in using both theories is to explore the conditions under which shear effects are significant or not by contrast. However, real applications impinge effects not covered by these simple models, such as stress concentration due to the force reactions on the support contacts, inhomogeneous temperature distribution, or structural instabilities, for example. Each of these effects leads to the violation of the planar cross sections assumption, hence resulting in a consequent modification on the deflection profile predicted by the EB or TE theories. Therefore, such a deviation may be considered as a measure of the presence and relevance of structural effects, although mixed. To individually distinguish whether a specific effect is on charge or not, one must tailor the boundary conditions or even the problem itself so that to favor or even ensure the absence of other effects besides the one of interest, since the deflection profile response is obtained as a whole, hence including all effects as if the system were a *black box*.

To obtain a reliable deflection profile of a simple beam under realistic conditions, one can either proceed with accurate experimentation, or simulate the system by means of numerical solution, such as the Finite Element Method (FEM). From the literature [7], depending on the complexity of the problem and degree of refinement, the main FEM software accomplish the actual structural behavior quite well; accordingly, we have assumed that the structural effects in general are implicitly accomplished too, or at least the most distinguishable ones. To circumvent experimental issues, we have chosen the numerical simulation option, and we have conceived the simplest case study possible so that to rely on the ANSYS software, [2].

Using the ANSYS output as realistic reference for the deflection profile, analytical solutions such as the ones from the EB or TE integer order differential equations can be evaluated on how much the fail to cover the structural effects. The greater the difference, the more relevant the sum of structural effects. However, the deviation on deflection alone may not be a good measurement of structural effects since this information depend on absolute values located on specific finite elements, thus directly depending on the numerical model resolution. It would be interesting to obtain another analytical and continuous solution from a differential equation already fed with the ANSYS data, hence containing the structural effects, so that to compare it with the EB or TE differential equations. One answer for that is to proceed with a fractional order differential equation obtained from the generalization of EB or TE differential equations (integer order) confronted against the ANSYS data. A fractional order value is then obtained for the resulting new fractional solution for each of the two cases, named here FEB and FTE. The new order can be evaluated on its significance by comparing it with the original integer order value. This difference constitutes a measure of relevance of the structural effects in a continuous and nonlinear differential equation, whose FEB and FTE solutions remain in good agreement with the actual deflection profile.

To obtain the individually consequent order deviation from the original integer value, each structure effect must be isolated in system implemented for the ANSYS simulation. To do so, an adequate modification is required, so that to ensure the same internal forces as the ones acting in the original problem accomplished by the EB and TE solutions. A separate assessment using fractional calculus is proposed in this work to, first, evaluate the effects of stress concentration on the support contacts, and, second, to prepare the modeling to potentially reveal another structural effect. The stress concentration is diagnosed from the ANSYS results, where its distribution is explicit. To evaluate its impact as a fractional order value, a thin beam is chosen so that to ensure no other effects, which is verified since the corresponding EB and TE solutions agree; therefore, the impact is the same, as no shear effects are present, which means that the solutions FEB and FTE are the same. In the sequence, to eliminate the presence of stress concentration, the aforementioned modification is proceeded in the ANSYS modeling: the beam is sufficiently extended in a symmetrical fashion till the absence of such a effect. Up to this point, all three solutions converge for the original beam interval of interest: EB, TE and ANSYS, and the fractional order is the same value of the integer order, which means that there is no structural effect at all. From now on, the problem is prepared to further investigations by means of modifications in the original problem, such as in the cross section height or width.

2 ANALYTICAL DEVELOPMENT

A simply supported beam is chosen for the investigation. The beam is prismatic, homogeneous, isotropic, linearly elastic, and L in length. The bending moment and shear force are represented by M and V, respectively. q is the uniformly distributed load along its length.

According to [6], BE beam theory states that

$$EI\frac{d^4v(x)}{dx^4} = -q, \tag{2.1}$$

where E is the Young's modulus and I is the rectangular moment of inertia of the cross section.

The model described in Eq.(2.1) is known as the Euler-Bernoulli equation, since the assumptions assumed in its development are equivalent to the EB theory, which is suitable for beams with a high spectrum ratio $\frac{L}{h}$, where L is the length of the beam and h is the height of its cross section. The hypotheses assumed in the EB theory are that cross sections initially flat, remain flat after deformation; the orthogonality of the cross sections in relation to the barycentric axis before the deflection is preserved, after the deflection.

For our specific problem, applying the boundary conditions $v(0) = v(L) = 0$ and $v'\left(\frac{L}{2}\right) = EIv''(0) = M(0) = 0$, the solution for the bending strain results

$$v(x) = \frac{q}{EI}\left(\frac{-x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right). \tag{2.2}$$

2.1 Fractional Euler-Bernoulli solution

In the 17th century, Newton and Leibniz independently created the integer order calculus, or calculus, where the concepts of integral and derivative merge in the famous fundamental theorem of calculus. From that time is also the arbitrary order calculus or non-integer order calculus, popularly coined with the name of fractional calculus, perhaps because it is related to a possible derivative with the order being a simple fraction [13].

In recent decades, applications using fractional calculus have advanced significantly in several areas of knowledge, such as physics, engineering, biology. In all these areas, the same is used to explain the behavior of complex phenomena that integer-order calculus cannot explain. Among many phenomena modeled by fractional calculus, we can cite those resulting from viscoelasticity, [12].

It is important to highlight that there are several types of approaches, in particular the type of formulation of non-integer order derivative, coined with the name of fractional derivative, [11, 18].

In this work, we use the formulation as proposed by Caputo that considers the fractional derivative as a fractional integral of an integer-order derivative [5] and [9], for example. In what follows we focus on solving a non-integer order differential equation. The fractional model proposed for such a solution is the fractional ordinary differential equation of order α, similar to the Eq. (2.1) , given by

$$\frac{d^{\alpha}v}{dx^{\alpha}} = -A, \tag{2.3}$$

with $A = \frac{q}{EI}$, $3 < \alpha \leq 4$ and $v = v(x)$.

For $m-1 \leq \alpha < m$ with α non-integer and m integer, the Laplace transform for derivatives of order α is given by [16]

$$L[f^{(\alpha)}(x)] = s^{\alpha}F(s) - \sum_{k=0}^{m-1}[s^{\alpha-1-k}f^{(k)}(0)], \tag{2.4}$$

where $F(s)$ is the Laplace transform of $f(x)$.

Applying Eq.(2.4) in Eq.(2.3), we have

$$s^{\alpha}V(s) - s^{\alpha-1}v(0) - s^{\alpha-2}v'(0) - s^{\alpha-3}v''(0) - s^{\alpha-4}v'''(0) \quad = -\frac{q}{EI}\,\frac{1}{s}. \tag{2.5}$$

From the boundary conditions, we have $v(0) = 0$. Thus, taking $v'(0) = k_1$, $v''(0) = k_2$ and $v'''(0) = k_3$, replacing in Eq.(2.5) and rearranging the terms, we get

$$V(s) = -\frac{q}{EI}\,\frac{1}{s^{\alpha+1}} + \frac{1}{s^2}k_1 + \frac{1}{s^3}k_2 + \frac{1}{s^4}k_3. \tag{2.6}$$

To obtain the solution in the x variable, the application of the inverse Laplace transform in Eq.(2.6) results in

$$v(x) = -\frac{q}{EI\,\Gamma(\alpha+1)}\,x^{\alpha} + k_1\,x + \frac{k_2}{2}\,x^2 + \frac{k_3}{6}\,x^3. \tag{2.7}$$

Since for the case study $v(L)=0$, $v'\left(\frac{L}{2}\right)=0$ and $EIv''(0)=M(0)=0$, from Eq.(2.7) we get

$$v(x)=\frac{q}{EI}\left[\frac{-x^{\alpha}}{\Gamma(\alpha+1)}+\frac{4L^{\alpha-3}}{2^{\alpha-1}\Gamma(\alpha+1)}(2^{\alpha-1}-\alpha)x^3+\frac{L^{\alpha-1}}{2^{\alpha-1}\Gamma(\alpha+1)}(4\alpha-3.2^{\alpha-1})x\right]. \tag{2.8}$$

Employing $\alpha=4$ into Eq.(2.8) and taking into account that for $n\in N$ we have $\Gamma(n+1)=n!$, the integer solution is recovered in Eq.(2.2).

In the vicinity of 4 at is right side, in the interval that matters, we solve the fractional differential equation

$$\frac{d^{\beta}v}{dx^{\beta}}=-B, \tag{2.9}$$

with $4<\beta\leq 5$ and B a positive constant. In this case, we consider the same boundary conditions as in the previous problem, in addition to the fifth condition that naturally arises, when we impose constraints to get the solution of interest. Finally, we get the solution of interest taking the limit $\beta\longrightarrow 4$. The solution obtained in this neighborhood is the same solution as given Eq.(2.8).

To obtain the fractional bending moment, we can derive Eq.(2.8) twice and multiply by EI. By doing this, we obtain

$$M(x)=\frac{-q\,(\alpha-1)}{\Gamma(\alpha)}\,x^{\alpha-2}+k_2\,EI+k_3EIx. \tag{2.10}$$

From the development used to obtain Eq.(2.10) and applying the boundary conditions $M(0)=M(L)=0$, the fractional solution of the bending moment results

$$M(x)=\frac{q(\alpha-1)}{\Gamma(\alpha)}\left(-x^{\alpha-2}+L^{\alpha-3}x\right). \tag{2.11}$$

For $\alpha=4$,

$$M(x)=\frac{-qx^2}{2}+\frac{qL}{2}x.$$

Analogously, we obtain the expression for the modified (fractional) transversal shear force. Employing the boundary condition $V\left(\frac{L}{2}\right)=0$ and using the fact that the derivative of the bending moment is the shear force, we obtain

$$V(x)=\frac{q(\alpha-2)(\alpha-1)}{\Gamma(\alpha)}\left(-x^{\alpha-3}+\frac{L^{\alpha-3}}{2^{\alpha-3}}\right). \tag{2.12}$$

Also for $\alpha=4$, the original integer order expression for the shear force is recovered, as follows

$$V(x)=-qx+\frac{qL}{2}.$$

2.2 The shear effects of the Timoshenko-Ehrenfest beam theory

From the TE beam theory [3], for cross sections suffering small distortions, without losing linearity, the shear deformation is given by the first order ordinary differential equation

$$\frac{dv}{dx}=-c\frac{V}{GA}, \tag{2.13}$$

where $v = v(x)$, $V = V(x)$, G is the cross elasticity modulus, and c is the shear coefficient related to the bending deformation at the height of the section, which depends on its geometric dimensions and shape.

In Eq.(2.13), A is the cross-sectional area, $G = \frac{E}{2(1+\nu)}$ for rectangular sections (our case study), and $c = \frac{12+11\nu}{10(1+\nu)}$, where ν is a Poisson module.

Solving Eq.(2.13) considering V given in Eq.(2.12) for $\alpha = 4$, we obtain

$$v_c(x) = \frac{qc}{GA}\left(\frac{x^2}{2} - \frac{Lx}{2}\right). \tag{2.14}$$

Therefore, considering the bending and shear deformations, the final deflection can be given by means of the superposition principle, as below.

$$v(x) = v_f(x) + v_c(x), \tag{2.15}$$

where $v_f(x)$ and $v_c(x)$ are the bending and shear deformations, respectively in Eq.(2.2) and Eq.(2.14).

Thus,

$$v(x) = \frac{q}{EI}\left(\frac{-x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right) + \frac{qc}{GA}\left(\frac{x^2}{2} - \frac{Lx}{2}\right). \tag{2.16}$$

The characterization of the effects seen in this section are addressed by TE beam theory, discussed in the following.

The TE model considers, in addition to bending deformation, the shear effects that the beam can undergo when subjected to loading. In this case, the cross sections of a beam carry out, besides the translational displacement (deflection) and the rotation purely associated with the bending accomplished by the BE model, the rotation around their barycenter with respect to the neutral plane, in this model no longer kept orthogonal. This additional rotation comes from the shear stresses more present in the extremities of thicker beams [3] and [8]. In essence, the first premise of EB's theory is preserved, but not the second. The resulting system of differential equations is

$$\begin{cases} \frac{d^3\theta}{dx^3} = \frac{q}{EI}, \\ \frac{dv}{dx} = \theta - \frac{EIc}{GA}\frac{d^2\theta}{dx^2}, \end{cases} \tag{2.17}$$

where the dependent variables of the $\theta = \theta(x)$ and $v = v(x)$ system represent, respectively, the cross section rotation and the deflection of the barycentric axis, therefore including the shear effects.

Solving Eq.(2.17) for the particular case of a simply supported under uniformly distributed load, the deflection results

$$v(x) = \frac{q}{EI}\left(-\frac{x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right) + \frac{cq}{GA}\left(\frac{x^2}{2} - \frac{Lx}{2}\right) \tag{2.18}$$

for the boundary conditions $\theta\left(\frac{L}{2}\right) = 0$, $M(0) = M(L) = 0$ and $v(0) = 0$.

Since Eq.(2.18) matches Eq.(2.16), we propose in this work a fractional solution for the TE model.

2.3 Fractional Timoshenko-Ehrenfest solution

Using the superposition principle as stated in Eq.(2.16), we propose a fractional solution for the TE equation as follows:

$$v(x) = v_{FEB}(x) + v_{FC}(x), \tag{2.19}$$

where $v_{FEB}(x)$ and $v_{FC}(x)$ are the fractional solutions for deflection (effects of bending moments) and shear, respectively.

The solution $v_{FEB}(x)$ was obtained in Eq.(2.8). On the other hand, we obtain $v_{FC}(x)$, integrating Eq.(2.13), using the fact that the shear force integral V is the bending moment M, and finally we consider M as the fractional bending moment given in Eq.(2.11). From that, we get

$$v_{FC}(x) = \frac{c}{GA}\frac{q(\alpha-1)}{\Gamma(\alpha)}\left(x^{\alpha-2} - L^{\alpha-3}x\right). \tag{2.20}$$

Finally, taking Eq.(2.20) and Eq.(2.8) in Eq.(2.19), we have

$$v(x) = C\left[-x^{\alpha} + \frac{4L^{\alpha-3}}{2^{\alpha-1}}(2^{\alpha-1}-\alpha)x^3 + \frac{L^{\alpha-1}}{2^{\alpha-1}}(4\alpha - 3.2^{\alpha-1})x\right] + D\left(x^{\alpha-2} - L^{\alpha-3}x\right), \tag{2.21}$$

where $C = \frac{q}{EI\Gamma(\alpha+1)}$ and $D = \frac{cq(\alpha-1)}{GA\Gamma(\alpha)}$.

The equation (2.21) is the fractional TE solution that provides the deflection suffered by a simply supported beam under an uniformly distributed load.

3 APPLICATION AND RESULTS

Confronting Eq.(2.8) and Eq.(2.21) against the ANSYS results, the α values for the particular solutions of FEB and FTE, respectively, can be found. For the sake of clarity, we have conceived a simple example of structure loaded within the linear elastic limits: a simply supported AISI 1020 steel beam, isotropic, prismatic, with dimensions $(b,h,L) = (0.3;0.8;5)$, and subjected to an uniformly distributed transversal load $q = 1\times10^4 N/m$. According to ANSYS simulation, the maximum stress for such a load is $\sigma_{max} = 3.1996\times10^6 Pa$, hence far below the maximum yield stress for this material, on average $\sigma_{max} = 4.2\times10^8 Pa$ [19].

As the variable in question is the transversal deflection, from symmetry, the maximum deflection locates in the middle of the beam, i.e. $x = 2.5m$. We have chosen this position to obtain the fractional order α because of its relevance to prevent failure. From ANSYS, the maximum deflection is $3.5609\times10^{-5}m$. Then, the resulting outcome is $\alpha = 4.1585$ for FEB and $\alpha = 4.0876$ for FTE, as expected, as the closer α is to 4, the less influential is the fractional order “needed” for adjustment toward the problem under investigation (accurately modeled in ANSYS). This means that the original integer order TE model is confirmed closer to the actual real structure behavior, since the interference from the fractional adjustment in FTE is lower than for the case of the original integer order EB model; this is coherent with the fact that the TE model encompasses the shear effects, on the contrary of EB, and represents better the real structure behavior. Employing these α values back into Eq.(2.8) and Eq.(2.21), correspondingly, the particular fractional solutions are achieved. From this point onward the final equations for deflections can be compared with the ANSYS response along the entire beam.

The graph in Figure 1 reveals the deflection behavior of EB, FEB, TE and FTE solutions with respect to ANSYS.

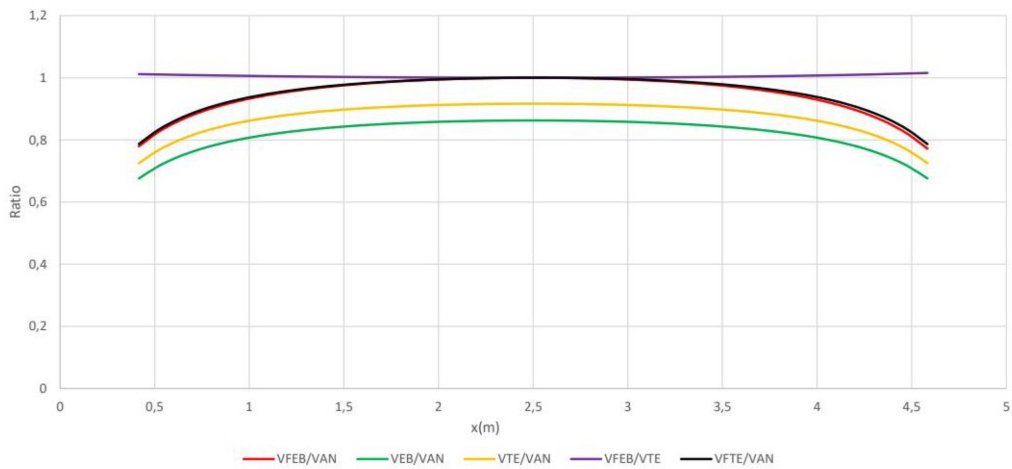

Figure 1: Deflection ratio for the solutions of the integer and fractional order with respect ANSYS, and deflection ratio for the solutions of FEB with respect to TE.

Besides, we have included the deflection ratio between FEB and TE solutions (in this case, $\alpha = 4.0614$). All five graphs are symmetrical and show progressively larger differences from the middle to each extremity, which is in accordance with the shear effects, that lead the structure to an increased transversal deflection to accommodate those shear deformations. For this reason in the ANSYS model, the closest to reality, the deflection is the highest; almost as much as it would be in an experiment, all structural relaxation tend to be contemplated in the numerical model. Still with respect to ANSYS, from the worst to the best are the graphs EB, TE, FEB and FTE, behaving as expected. EB lacks on the shear effects, and that is why it is significantly worse (far from ANSYS) than TE; on the other hand, when comparing FEB and FTE, the original shear effects consideration in TE still makes FTE prevail over FEB, but very slightly. The latter shows that the fractional calculus approach is capable of achieving a good model adjustment not depending importantly on the accuracy of the original integer order model (at least for this application). Besides, from the comparison between FEB and TE, as the first is closer to ANSYS, FEB surely incorporates the shear effects originally absent in EB, and as the results suggest, possibly other physical effects.

From this initial analysis, the first relevant fact to be observed is that we have the FEB fractional solution, contemplating the shear effects, previously only compensated by the TE solution. To better characterize this observed fact, let us consider another thick beam, made of AISI 1020 steel, with dimensions $(b,h,L) = (0.3m;0.9m;5m)$, submitted to a distributed loading with module given by $q = 3\times 10^4 N/m$. Note that the beam is thick, as its spectrum ratio $\left(\frac{L}{h} = 5.5555\right)$ and therefore, the shear effects influence in the deflection, as in experiments done by [14], these effects are considerable for a spectrum ratio less than 10, which is equivalent to a maximum displacement greater than or equal to 2% with respect to the maximum displacement of the deflection EB. In our case study, we have that the difference between the maximum deflections TE and EB is equal to 7%.

Confronting the maximum solutions of the FEB and TE equations, we have that the maximum solution FEB is taken in the maximum solution TE to $\alpha = 4.0779$. Taking such α into the generalized FEB equation, we obtain the particular FEB solution, which includes the shear effects.

The graph in Figure 2, confronts the FEB solution for $\alpha = 4.0779$ with the TE. It can be seen that the FEB solution includes the TE solution throughout the interval in which the beam is defined.

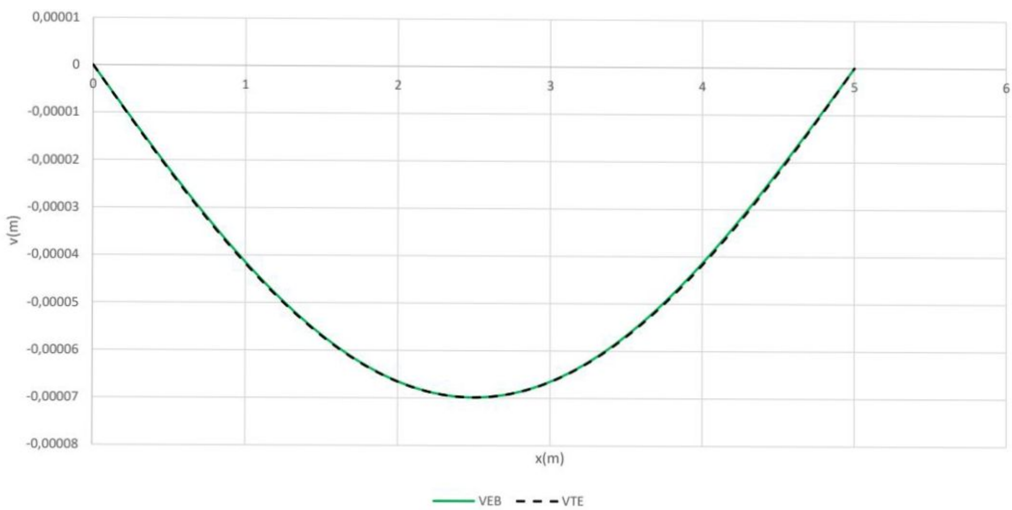

Figure 2: Confrontation between FEB and TE solutions.

To better characterize this fact, we determined the relative error E_r between the FEB solutions with $\alpha = 4.0779$ and TE, defining E_r as being

$$E_r = \frac{|v_{TE} - v_{FEB}|}{|v_{TE}|}, \tag{3.1}$$

where v_{TE} is the deflection corresponding to the solution TE and v_{FEB} is the deflection FEB for $\alpha = 4.0779$.

Finally, we calculate the average error E_m, defined as the arithmetic average of the elements of the vector generated by Eq.(3.1), obtaining $E_m = 0.6000\%$. We have an error less than 1% between the respective solutions. Therefore, we can state that the FEB solution for the aforementioned α is not identical to the TE solution, but presents an expressive approximation in the entire domain in which the beam is defined.

3.1 Study of the effects contemplated by the FEB and FTE solutions

In the referred application of Section 3, we highlight that, in addition to the FEB and FTE solutions, contemplating the shear effects, they present a better approximation than the entire TE solution, when compared with the ANSYS numerical solution. Therefore, we can conclude that these solutions are compensating for other physical effects. So, continuing the research, our first desired fact was to characterize these effects that are being contemplated by such solutions.

Searching the literature, we found that in simply supported beams, especially thick ones, the stresses present in the supports are much more intense than the stresses produced by bending or shear moments, [19]. These stress concentrations can cause the beam deflection to vary throughout its domain, even when the beam is subjected to low stresses, [10].

So, in our problem, analyzing its spectrum ratio, we have $\left(\frac{L}{h} = 6.2500\right)$, being equivalent in this case to a difference of 6% between the respective maximum deflections EB and TE. Therefore, we conclude that the beam used is thick. As the values the α that lead to the maximum FEB and FTE solutuion, in the solution provided by the linear and nonlinear simulations (which are relatively close, in this case, as they present average error between their respective deflections given by $E_m = 0.0020\%$), are greater than the entire order, so we have that the referred physical effect is contributing to the ANSYS simulations presenting greater deflections than the TE solution.

Analyzing the stresses in the respective ANSYS simulations, we see that the concentration of maximum stresses are in the supports, as shown in Figure 3.

Therefore, our hypothesis is that the physical effect, which is altering the deflection, in these simulations is caused by the supports.

To confirm this hypothesis, we thought of a way to carry out the simulation in ANSYS, without the influence of the effects of the supports. For this, we consider a $25m$ beam as shown in Figure 4.

The beam is simply supported with a distributed load q across its span, supports at points A and B, whose coordinates are $6m$ and $19m$, respectively. Our objective here is to compare with the analytical solution TE, the deflection obtained in the central 5 meters, that is, between the points C and D of the problem implementation in ANSYS, of a beam, whose dimensions are $(b,h,L) = (0.3m; 0.8m; 25m)$. The respective points are $4m$ from the supports, aiming to filter out such effects. If the compared solutions converge, our hypothesis is proven.

The intervals corresponding to the segments $\overline{EA} = \overline{BF} = 6m$ are necessary so that we have zero bending moment at points C and D and then, we can compare the solution obtained from the ANSYS implementation with the analytical solution TE.

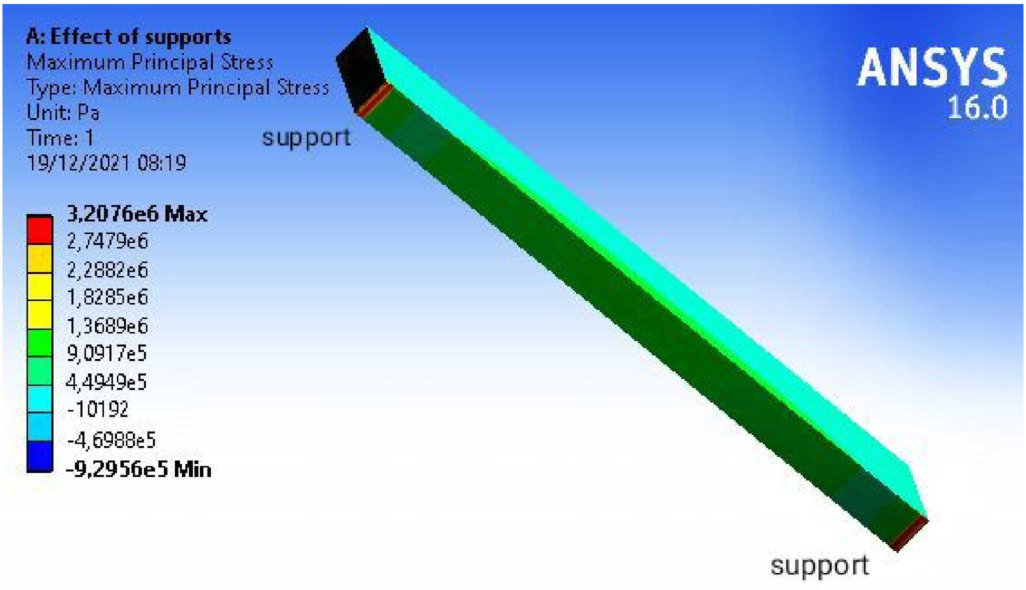

Figure 3: Concentration of stresses.

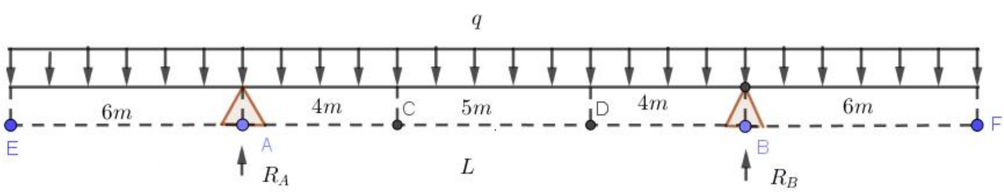

Figure 4: 25 meter beam, simply supported with distributed load.

Implementing the ANSYS, taking the deflection in the interval defined by the central $5m$ and calculating the average error between the deflection obtained in ANSYS and the TE deflection for the beam, whose dimensions are $(b,h,L) = (0.3m; 0.8m; 5m)$, we get $E_m = 0.4600\%$. Therefore, the solutions after filtering the supports, present a significant approximation, thus ensuring the proof of our hypothesis.

The graph in Figure 5 shows the respective graphs of TE solutions and ANSYS solutions: a) without filter and b) with filter, respectively.

Therefore, as shown, the physical effects are from the supports on the deflection, which are covered by the generalized fractional solutions FEB and FTE, with relevant proximity to the real situation, which we are representing by the ANSYS numerical solution. Furthermore, as the order α is not integer, we also have, in the fractional order, the identification of the nonlinearity of the phenomenon.

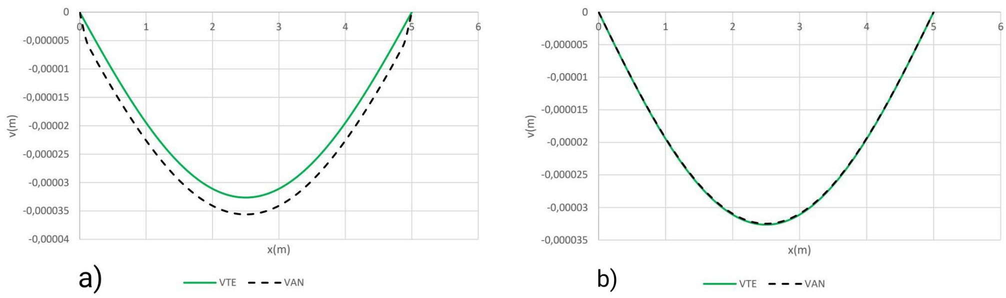

Figure 5: TE solution compared to ANSYS without filter and with filter.

The fact that fractional solutions detect the effects of supports on the deflection is extremely important, as this is not covered by the integer solutions EB and TE, as the models from which they result are developed considering the undeformed condition of the structure, [1] and [15]. Hence, we can highlight that these fractional models allow us, from models deduced under undisturbed conditions, to generalize them, making them capable of evaluating phenomena that can only be studied, considering the deformed condition of the structure.

4 SLENDER BEAMS AND THE EFFECTS OF SUPPORTS

In this section we analyze the slender cases where the beam is subjected to small and large deflections, respectively.

We have already done a case study for a slender beam subjected to low stresses in [11] and we proved that in such a configuration, there is no influence of the support effects, as the solutions EB, TE and the ANSYS simulations are linear and nonlinear converge.

We will do another case study here, considering the beam of dimensions $(b,h,L) = (0.05m;0.1m;5m)$, whose spectrum ratio is $\frac{L}{h} = 50$, which is the same as the beam studied in [11] but with a much smaller cross section basis. Our objective is to show better that for these types of beams, the supports do not significantly influence the deflection, where we will use the methodology, already used in Subsection 3.1, that is, the support filtering process.

Performing linear and nonlinear simulations with load $q = 10^2 N/m$, whose yield stress is $\sigma = 3.7504 \times 10^6 Pa$, for both simulations and therefore within from the linear limits, we see that the deflections converge, because calculating the average error between the deflection of the respective simulations, we obtain $E_m = 0.0020\%$. Furthermore, we found that the EB, TE and ANSYS solutions also converge, as shown in Table 1.

Due to the coincidence between the respective solutions, as seen in Table 1, we conclude that there are no nonlinear physical effects present in this beam configuration, as the only physical effects present are due to the variation of the bending moment. Thus, we can conclude that there is no influence of supports, even without using our methodology for the filtering process. Fur-

Table 1: Average error between solutions.

compared deflections	average error E_m
EB-TE	0.1000%
EB-AN	0.1030%
TE-AN	0.0162%

thermore, the solutions EB and TE, that is, the generalized fractional solutions for the particular $\alpha = 4$, contemplate such effects.

On the other hand, when the influence of the supports is not significant, an increase in stresses is noticed near the central and lower region of the beam. Based on these observations, we decided to investigate what happens to the deflection if we take loads, which cause stress variation in the beam close to the yield limit.

For this, we take the beam with the same dimensions as the previous case subjected to load $q = 5 \times 10^4 N/m$. Performing the linear and nonlinear simulations in ANSYS and analyzing the stress distribution in the beam, we noticed that there is an increase near the central and lower region of the beam in both simulations, as shown in Figure 6.

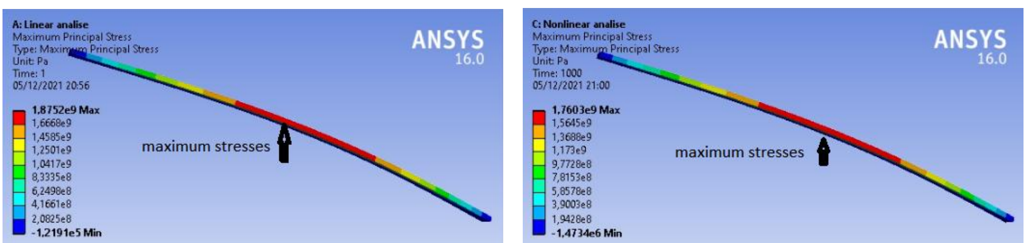

Figure 6: Stress concentration in linear and nonlinear simulations.

Assessing the stresses present in the beam represented in the Figure 6, it is noticed that the values in the nonlinear simulation are smaller than the respective values in the linear simulation. To better characterize this fact, we take the average stresses of both simulations, obtaining $\sigma = 1.2300 \times 10^9 Pa$ and $\sigma = 1.1600 \times 10^9 Pa$, respectively. Therefore, we can conclude that the stresses in the nonlinear simulation decreased. From this confirmation, we expect that the deflection in the nonlinear simulation will also be smaller than the deflection in the linear simulation.

To verify this, we take the average deflection for the respective cases, analogous to what was done in the case of stresses. From this analysis, we obtain $f_{ml} = 0.2959$ and $f_{mnl} = 0.2765$, where f_{ml} and f_{mnl} are the average deflections for the linear and nonlinear, respectively. Hence, we find that the nonlinear deflection is actually smaller than the linear deflection, so the nonlinear simulation is contemplating a physical effect not covered by the linear simulation. We affirm that these effects are not caused by the supports, because as seen in Figure 6, the greatest variation of stresses in the beam is not close to the supports, but close to the central region of the beam.

To substantiate our claim, we perform the filtering process again, as done in the thick case study. For this, we will implement in ANSYS the beam, whose dimensions are $(b,h,L) = (0.05m; 0.1m; 25m)$ with the same settings shown in Figure 4 and compare the deflection obtained in the central $5m$ of the linear and nonlinear simulations with the analytical solution TE for the beam $(b,h,L) = (0.05m; 0.1m; 5m)$. For our claim to be proven, the ANSYS nonlinear simulation must not converge to the TE analytical solution.

From the respective simulations performed, we can conclude that the linear EB, TE and ANSYS solutions converge, but as expected, the nonlinear ANSYS simulation solution does not converge to the analytical TE solution. This guarantees the validity of our claim.

In Table 2, we represent the average error E_m obtained when we confront the deflections of each analysis.

Table 2: Average error between solutions.

compared deflections	average error E_m
EB-TE	0.1006%
TE-ANSYS linear	0.0090%
TE-ANSYS nonlinear	18.0616%

The graphic in Figure 7 shows, after the filtering process, the comparison between: a) the TE analytical solution and ANSYS linear simulation and b) the TE analytical solution and nonlinear ANSYS simulation. From this, we have a visual representation that the physical effect detected in the nonlinear simulation has no influence from the supports.

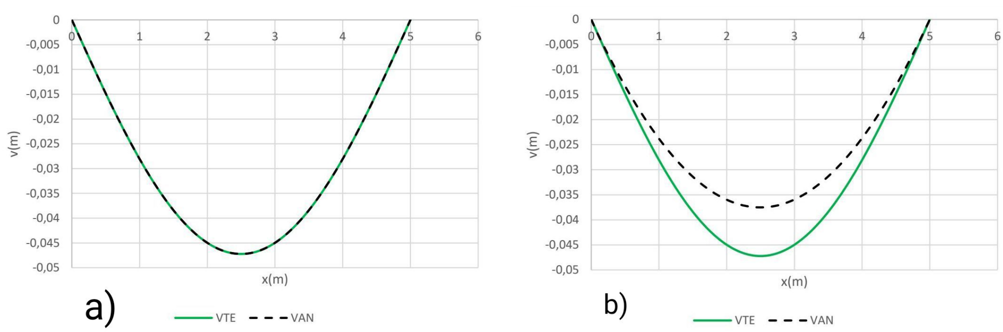

Figure 7: Comparison between the graph of the TE analytical solution and the ANSYS solution.

Having proven that such a physical effect is not caused by supports reactions, we searched the literature for reports on its cause. Hence, we find similar numerical simulations done by [1]. According to that reference, this effect is a consequence of the axial forces that arise from the tendency to approach the ends of the beam, as it is being subjected to loads that cause large deflections. Due to the non-linearity of the phenomenon, the axial forces that arise are not proportional to the displacements, so there is an increase in stiffness, caused by the stiffening due to the traction of the lower beam fibers, [15]. This fact justifies the increase in stresses observed near the central and lower region of the beam, as shown in Figure 6. In this case, the variation is in geometric stiffness. Due to this increase in stiffness, we identified the reduction in deflection of the nonlinear simulation when compared to the deflection of the linear simulation.

Proceeding similarly to what was done in the initial application of Section 3, we can evaluate the interpretation of such effects by the fractional analytical solutions FEB and FTE.

Confronting the respective fractional solutions with the ANSYS solution, we see that the maximum deflection of the FEB and FTE fractional solutions are taken into the maximum deflection of the ANSYS solution when we take $\alpha = 3.8040$. Taking the value of α in the generalized solutions we find the particular solutions FEB and FTE for the deflection in the entire interval in which the beam is defined.

Finally, calculating the ratios $\frac{v_{EB}}{v_{AN}}$, $\frac{v_{TE}}{v_{AN}}$, $\frac{v_{FEB}}{v_{AN}}$ and $\frac{v_{FTE}}{v_{AN}}$, we get the graph of Figure 8.

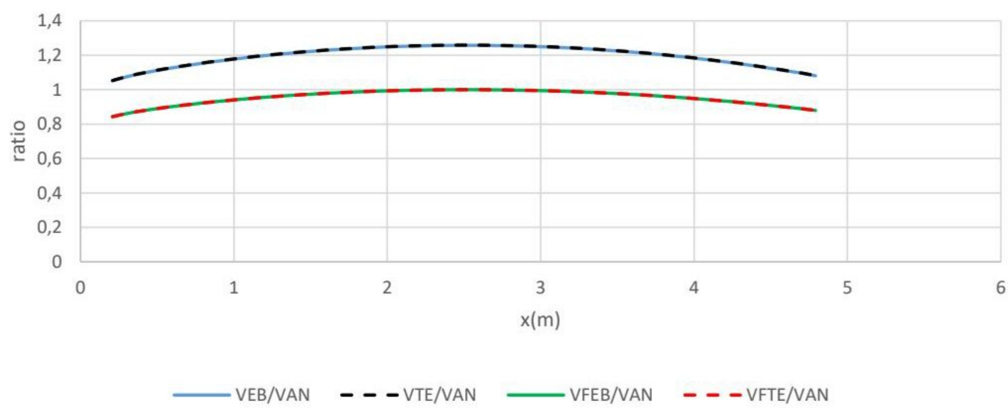

Figure 8: Ratio between fractional solutions EB, TE, FEB and FTE with respect to ANSYS solution.

Analyzing the graph in Figure 8, we can see that the fractional analytical solutions FEB and FTE include the ANSYS solution in much of the interval in which the beam is defined, with the best approximation occurring in the central part of the beam. In general, such solutions present very significant approximations of the ANSYS numerical solution, if compared to the EB and TE solutions. Therefore, they contemplate the physical effect considerably, in addition to characterizing its nonlinearity as well, as the α that adjusts the solution is different from the entire order.

We conclude by emphasizing that the non-integer order calculus can add more information to the classical calculus, making the description of natural phenomena more accurate. Therefore, it comes from this fact, the relevance of the FEB and FTE analytical fractional solutions in the interpretation of the physical phenomena studied in this paper, which, as proven through the study of analyzed cases, present a significant approximation to the real situation (represented by the numerical solution obtained, using the software ANSYS). As we have already emphasized, the phenomena interpreted in this paper are nonlinear and therefore are not covered by the integer analytical solutions EB and TE.

The characteristic present in these fractional solutions that allow interpreting phenomena beyond linear phenomena is the memory effect carried by α, with $\alpha \neq 4$, which is the entire order in the case studied. Interesting discussions of the memory effect are presented in [4, 17].

5 CONCLUSIONS

In this work, we propose a new measure for structural effects that alter the deflection profile provided by the Euler-Bernoulli (EB) and Timoshenko-Ehrenfest (TE) for simply supported beam subjected to uniformly distributed static load. Amongst them, the stress concentration due to the force reactions on the support contacts, focus of this research. The reference for this measure is the ANSYS software, an accurate and reliable source of realistic information concerning the structural behavior, which implicitly encompass all effects in general as a whole; the simplicity of the system enhances potential of accuracy and analysis. To evaluate such effects, a separate assessment to the causes of deflection profile alteration is newly proposed in this work. Preserving the beam interval of interest, the ANSYS modeling is adjusted in accordance. Proceeding with fractional solutions for EB and TE, here called FEB and FTE, the resulting fractional order becomes a new measure for structural effects; is this particular, for the stress concentration. The employment of EB, TE, FEB and FTE serve the purpose of evaluating the shear effects as well, besides to proceed with the separate assessment investigation. As a final result of this work, the problem has been explored till its preparation for further investigations towards additional structual effects.

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REFERENCES

[1] F.A. Alves. “Finite Elements: a base of CAE technology: Nonlinear analysis”. Érica, São Paulo, 1st ed. (2012).

[2] ANSYS, INC. Release. “17.0 documentation for Ansys”. Canonsburg (2016).

[3] A.E. Assan. “Strength of Materials”. Editora Unicamp, Campinas, SP (2010).

[4] L.C. Barros, M.M. Lopes, F. Santo Pedro, E. Esmi, J.P.C. dos Santos & D.E. Sanchez. The memory effect on fractional calculus: an application in the spread of COVID-19. *Computational and Applied Mathematics*, **40**(3) (2021), 1–21.

[5] R.F. Camargo & E.C. de Oliveira. “Fractional Calculus”. Editora Livraria da Física, São Paulo (2015).

[6] H.R. Charles. “Strength of Materials”. Pearson Educaciòn, São Paulo (2004).

[7] J.P.S. de Oliveira. “New proposition for checking the limit state of lateral buckling with distortion of composite steel and concrete beams”. Ph.D. thesis, UFMG, Universidade Federal de Minas Gerais, Belo Horizonte, MG (2018).

[8] S.A. Fleishfresser. “A formulation of the boundary element method for the analysis of Timoshenko beams”. Ph.D. thesis, UFPR, Universidade Federal do Paraná, Curitiba, PR (2012).

[9] R. Herrmann. “Fractional Calculus: An Introduction for Physicists”. World Scientific, GigaHedron, 2nd ed. (2014).

[10] X.F. Li & K.Y. Lee. Effect of horizontal reaction force on the deflection of short simply supported beams under transverse loadings. *Journal of Mechanical Sciences*, **99** (2015), 121–129.

[11] J.I.S. Lustosa, F.C. Bannwart & E.C. de Oliveira. Fractional calculus applied to evaluate stress concentration and shear effects in simply supported beams. *Revista Eletrônica Paulista de Matemática*, (2022).

[12] F. Mainardi. “Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models”. World Scientific, London (2010).

[13] B. Ross. “A brief history and exposition of the fundamental theory of fractional calculus”. Springer, Berlin (1975).

[14] T.C. Silva. Analytical study of Bernoulli and Timoshenko beam theories for varied boundary conditions. *Revista de Engenharia Civil IMED*, **6**(1) (2019), 57–70.

[15] H.L. Soriano. “Finite Elements: Formulation and Application on Statics and Dynamics of Structures”. Editora Ciência Moderna, Rio de Janeiro (2009).

[16] G. Teodoro. “Fractional calculus and the Mittag-Leffler functions”. IMECC, Unicamp, Campinas, SP (2014), 1-89 p. Dissertação de Mestrado.

[17] G. Teodoro, D.S. Oliveira & E.C. de Oliveira. On fractional derivatives. *Revista Brasileira de Ensino de Física*, **40**(2) (2017).

[18] G.S. Teodoro, J.A.T. Machado & E.C. de Oliveira. A review of definitions of fractional derivatives and other operators. *Journal of Computational Physics*, **388** (2019), 195–208.

[19] S.P. Timoshenko & E.G. James. “Mechanics of Solids”. LTC-Livros Técnicos e Científicos, Rio de Janeiro, 1st ed. (1983).