

On the Exact Boundary Control for the Linear Klein-Gordon Equation in Non-cylindrical Domains

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ABSTRACT. The purpose of this paper is to study an exact boundary controllability problem in non-cylindrical domains for the linear Klein-Gordon equation. Here, we work near of the extension techniques presented by J. Lagnese in [12] which is based in the Russell's controllability method. The control time is obtained in any time greater than the value of the diameter of the domain on which the initial data are supported. The control is square integrable and acts on whole boundary and it is given by conormal derivative associated with the above-referenced wave operator.

Keywords: exact boundary controllability, non-cylindrical domains, linear Klein-Gordon equation.

1 INTRODUCTION

Since the second half of the last century to now we have seen an increase in the interest of the mathematicians and engineers in the study of vibratory problems modeled by the wave equation. This can be confirmed by great volume of the works dealing with such problems presented in the literature. In fact, such problems have theoretical and practical importance because they have wide applications in many branches of the engineering and mathematics.

An interesting part is to study vibratory problems that occur on a flexible body whose boundary deforms over time. Such phenomena generate, on the space-time scale, the non-cylindrical domains. An illustrative example is a metallic body in vibration that is placed in an environment subject to a change in the temperature. The increasing or decreasing of the temperature causes a linear expansion or contraction of the metallic body characterizing a movement of its boundary along of the time generating a non-cylindrical domain.

In the literature we can find many works that study the most diverse types of phenomena involving wave equations over non-cylindrical domains. Problems involving the existence of solutions,

energy decay, stabilization and control of wave equations on non-cylindrical domains have already been considered by several authors, to cite a few, see [1,2,3,4,5,6,7,8,12,13,14,15,18]. Especially the problems that deal with control processes for wave equations have been extensively studied in the last 50 years. We can find in the literature many works that brought a great contribution to the development of the control theory for hyperbolic equations. In highlight we can cite the papers ([3,13,18]) which established important methods for the study of controllability of wave equations, but they have worked only on cylindrical domains.

When we look specifically at control problems for waves equations in non-cylindrical domains we realize that, in the literature, the number of these is very small when compared to control problems for waves in cylindrical domains. But this seems natural, since the work on cylindrical domains is simpler than on non-cylindrical domains and often to study control problems in non-cylindrical domains we need, at some point, to resort to the techniques developed to treat controllability problems on the cylindrical domains. However, we can find interesting papers in the literature that deal with exact controllability for wave equations in non-cylindrical domains with the most diverse types of controllability methods, to cite a few see [2,4,7,8,12,14,15] and citing papers.

Taking into account the importance of the theme, this work proposes to study, in the light of the ideas of [12], an exact boundary controllability problem for the linear Klein-Gordon equation in non-cylindrical domains. In order to establish the results let us consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded and smooth by parts domain whose value of the its diameter will be denoted by $d(\Omega)$, and c a positive real number. Let Q be a non-cylindrical open set in $\mathbb{R}^N \times [0, +\infty)$ such that the intersection of Q with hyperplane $\{(x, t) \in \mathbb{R}^{N+1}, t \geq 0\}$ is a nonempty bounded open set Ω_t in \mathbb{R}^N and such that $\Omega_0 = \Omega$. We represent the boundary of Ω_t by $\partial\Omega_t$ and $\Gamma = \bigcup_{t \geq 0} \partial\Omega_t \times \{t\}$ is a variety that inherits from Ω the property of the smoothness by parts. Now, for $T \geq 0$, we set

$$Q_T = \bigcup_{0 \leq t \leq T} \Omega_t \times \{t\}, \quad \Gamma_T = \bigcup_{0 \leq t \leq T} \partial\Omega_t \times \{t\}.$$

In order to guarantee the well-posedness of the initial and boundary value problems to be considered we requires that Q_T , for $T \geq 0$, be contained in the conical time-like region

$$C = \bigcup_{\bar{x} \in \bar{\Omega}} \{(x, t) \in \mathbb{R}^N \times [0, +\infty); |x - \bar{x}|^2 \leq t^2\} \quad (1.1)$$

Being $U \subset \mathbb{R}^N$ an arbitrary domain, we denote by $L^2(U)$ and $H^1(U)$ the Lebesgue and Sobolev spaces, provided with theirs usual norms which will be denoted by $\|\cdot\|_{L^2(U)}$ and $\|\cdot\|_{H^1(U)}$ respectively (see [9]). Let us also consider the space $H_0^1(U)$ which is the closure of C_0^∞ in $H^1(U)$ provided with the norm of $H^1(U)$. The product space $H^1(U) \times L^2(U)$ is considered endowed with the product norm $\|(\cdot, \cdot)\|_{H^1(U) \times L^2(U)}^2 = \|\cdot\|_{H^1(U)}^2 + \|\cdot\|_{L^2(U)}^2$. The principals results of this paper are established in the two below theorems.

Theorem 1.1. Let $(f, g) \in H^1(\Omega) \times L^2(\Omega)$ and $\delta > 0$ be fixed. For every $T \geq d(\Omega) + \delta$, with exception of at most a finite number of them, there is a extension $(\tilde{f}_\delta, \tilde{g}_\delta) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ of (f, g) such that the solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + c^2 u = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R} \tag{1.2}$$

$$u(\cdot, 0) = \tilde{f}_\delta, \quad u_t(\cdot, 0) = \tilde{g}_\delta \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

vanishes in the finite cylinder $\bigcup_{d(\Omega)+\delta \leq t \leq T} \Omega \times \{t\}$, that is

$$u(\cdot, T) = u_t(\cdot, T) = 0 \quad \text{in} \quad \Omega. \tag{1.4}$$

The next theorem shows how the Theorem 1.1 can be used to obtain the exact boundary value control of the solutions of the linear Klein-Gordon equation in certain non-cylindrical domains.

Theorem 1.2. Let $(f, g) \in H^1(\Omega) \times L^2(\Omega)$ and $T > d(\Omega)$ be such that $\overline{\Omega}_T \subset \Omega$. Then there is a control function $h \in L^2(\Gamma_T)$ such that the solution of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + c^2 u = 0 \quad \text{in} \quad Q_T, \tag{1.5}$$

$$u(\cdot, 0) = f, \quad u_t(\cdot, 0) = g \quad \text{in} \quad \Omega_0, \tag{1.6}$$

$$v_t u_t - \nabla u \cdot v_x = h(\cdot, t) \quad \text{on} \quad \Gamma_T, \tag{1.7}$$

satisfies the final condition

$$u(\cdot, T) = u_t(\cdot, T) = 0 \quad \text{in} \quad \Omega_T. \tag{1.8}$$

Here (v_x, v_t) denotes the outward unit normal vector on the surface Γ_T at (x, t) point. The expression $v_t u_t - \nabla u \cdot v_x$ denotes the conormal derivative of u on Γ_T at point (x, t) . If Q_T were a cylindrical domain we have $v_t \equiv 0$, so the control function h would be determined by normal derivative of u . In (1.7) one has a time dependent boundary condition. Such conditions is very important in mathematical physics when dealing with diffraction problems for wave equations that need to be limited in a region of the space and some wave signs in the boundary acquire a velocity normal to its surface (see [10]).

We have mentioned above some papers that deal with different types of control problems for wave equations in non-cylindrical domains which use different controllability methods present in the literature. With respect the types of controllability problems the papers ([14], [7], [2]) have worked about internal exact controllability while the papers ([12], [8], [15], [4]) deal with exact boundary controllability. With respect the methods of controllability the references ([7], [8], [15]) have used the HUM Method establish in [13] for obtain its desirable controllability. The papers ([12], [4]) have worked near the Russell’s controllability method which is established in [18]. By the end in ([14], [2]) the authors obtain controllability by means of techniques based in terms of the Riemmannian metrics and geometric multipliers.

In the present paper we work in the light of Russell's controllability method in order to prove the Theorem 1.2 and also extension techniques showed in [12] in order to proof the Theorem 1.1. In the reference [12] the author used a extension technique, based in the Russell's controllability method, to obtain control for the standard wave equation in non cylindrical domains. As far as we know, there is no work in the literature that studies control for the Klein-Gordon equation in non-cylindrical domains using the methods as presented here. So, this is the contribution of this work. Finally, it is opportune to highlight the importance and usefulness of this method, because it is more easy to handle when dealing with domains that do not have a smooth geometry as proposed here, and this would be a difficulty to deal, for example, with the HUM method.

The rest of the present paper is organized in the following manner. The Section 2 is dedicated to do some essential preliminaries results. The Section 3 is devoted to prove the Theorem 1.2. The Theorem 1.1 is proved in the Section 4 and the paper is finalized with a references section.

2 SOME PRELIMINARIES

In this section we will highlight three important results that are essential in the proofs of the theorems proposed in the previous section. Such results are local energy decay estimates, analytic extension and suitable trace theorem to measure the regularity of the conditional derivative along of surfaces for the solution of the Cauchy problem to the Klein-Gordon equation.

2.1 Energy decay and analyticity

Let $U \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain and $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ initial data supported in U . Considering the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + c^2 u = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R} \quad (2.1)$$

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \quad \text{in} \quad \mathbb{R}^N. \quad (2.2)$$

It is known that the unique solution u of Cauchy problem (2.1)-(2.2) is such that $u \in H_{loc}^1(\mathbb{R}^N \times \mathbb{R})$ and it was established in [16] explicit formulas for the solution u when $t > \text{diam}(U)$ and $x \in U$.

Let $u(\cdot, t)$ be the solution of (2.1)-(2.2), of each $t > 0$ we define the operator solution $S_t : H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \longrightarrow H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ by $S_t(u(\cdot, 0), u_t(\cdot, 0)) = (u(\cdot, t), u_t(\cdot, t))$. The operator S_t applies the initial state $(u(\cdot, 0), u_t(\cdot, 0))$ into final state $(u(\cdot, t), u_t(\cdot, t))$ and for $t > d(U)$ such operator is compact, bounded and liner. In the proof of the Theorem 1.1 will be crucial the following result.

Lemma 2.1. *Let u be the solution of the Cauchy problem (2.1)-(2.2) with initial data $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ compactly supported in U and \mathfrak{R} the restriction operator to U . There exist positive real constants $T_0 > d(U)$ and K , independent of u_0 and u_1 such that*

$$\|\mathfrak{R}S_t(u_0, u_1)\|_{H^1(U) \times L^2(U)}^2 \leq \frac{K}{t^N} \|(u_0, u_1)\|_{H^1(U) \times L^2(U)}^2 \quad (2.3)$$

for every $t \geq T_0$.

The proof of the above lemma is obtained by a direct manipulation of the Theorem 2.1 of [16].

Considering still $u(.,t)$ as the solution of the Cauchy problem (2.1)-(2.2) with initial state $(u(.,0), u_t(.,0)) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ compactly supported in U , another essential element for the proof of the Theorem 1.1 is the analytic extension of the map $t \mapsto (u(.,t), u_t(.,t))$ to the sector $\Sigma_0 = \{\zeta = T_1 + z, |\arg(z)| \leq \pi/4\}$ as proved in [17]. For our purpose, such result can be adapted in terms of the operators S_t in the next lemma.

Lemma 2.2. *Let $u(.,t)$ be the solution of Cauchy problem (2.1)-(2.2) with initial state $(u(.,0), u_t(.,0)) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ compactly supported in U and considers the operators S_t as defined above. Then the family of compact linear operator $\{S_t : t > d(U)\}$ extends analytically to a family of linear compact operators $\{S_\zeta : \zeta \in \Sigma_0\}$, where Σ_0 is complex sector $\{\zeta = T_1 + z, |\arg(z)| \leq \frac{\pi}{4}\}$, being T_1 any constant greater than $d(\Omega)$. The proof of the Lemma 2.2 is an immediate consequence of the Theorem 1.1 of [17] rewritten it in terms of operators S_t .*

2.2 Trace regularity

In this part we express a result on the regularity of the traces of the solution of the Klein-Gordon which it is essential to proof the Theorem 1.2. Let us begin take account some notations and definitions. Let $P(\xi, D)$ be a linear second order hyperbolic partial differential equation with C^∞ coefficients depending on ξ in some open bounded domain $\Xi \subset \mathbb{R}^N$. Being $\Sigma \subset \Xi$ an oriented smooth hypersurface which is time-like and non-characteristic with respect to $P(\xi, D)$. Let $\eta = (\eta_1, \dots, \eta_N)$ be a unit normal to Σ . If $\sum a^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j}$ is the principal part of $P(\xi, D)$, then the expression $\frac{\partial u}{\partial \eta} = \sum a^{ij} \frac{\partial u}{\partial \xi_i} \eta_j$ defines the conormal derivative of u relative to the $P(\xi, D)$ along Σ . An important fact it is to know what the regularity of the traces of the conormal derivative on surfaces, for this purpose we turn to the paper [19]. Considering $\Xi \subset \mathbb{R}^N$, with $N \geq 2$, the Theorem 2 of [19] proves that if $u \in H^1_{loc}(\Xi)$ is such that $P(\xi, D)u \in L^2_{loc}(\Xi)$ then $\frac{\partial u}{\partial \eta} \in L^2_{loc}(\Sigma)$.

Particularly, if we consider $P(\xi, D)$ as being the Klein-Gordon operator, so its principal part will be $\frac{\partial^2}{\partial t^2} - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. Now, if γ is a smooth hypersurface in \mathbb{R}^N let us consider the surface $\gamma \times \mathbb{R}$ whose the unit normal vector is $v = (v_x, v_t)$ where $v_x = (v_1, \dots, v_N)$. In this case the conormal derivative of u along $\gamma \times \mathbb{R}$ is $\frac{\partial u}{\partial v} = v_t u_t - \nabla u \cdot v_x$. Particularly, if we apply the trace result mentioned in the previous paragraph for the Klein-Gordon operator we obtain the following result.

Lemma 2.3. *Let u be the solution of the Cauchy problem (2.1)-(2.2) with initial data $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Let γ be a smooth hypersurface in \mathbb{R}^N , with $N \geq 2$, with no self intersection and considers the surface $\gamma \times \mathbb{R}$ which the unit normal vector is $v = (v_x, v_t)$. Then the conormal derivative of u along $\gamma \times \mathbb{R}$ has trace $v_t u_t - \nabla u \cdot v_x \in L^2_{loc}(\gamma \times \mathbb{R})$.*

3 PROOF OF THE THEOREM 1.2

Choose $\delta > 0$ and $T \geq d(\Omega) + \delta$ such that $\bar{\Omega}_T \subset \Omega$. Given $(f, g) \in H^1(\Omega) \times L^2(\Omega)$, take $\bar{T} \in [d(\Omega) + \delta, T]$ for which exists an extension $(\tilde{f}_\delta, \tilde{g}_\delta)$ according Theorem 1.1, and let $\tilde{u} \in H^1_{loc}(\mathbb{R}^N \times [0, +\infty))$ be the solution to the Cauchy problem (1.2)-(1.3) with initial data $(\tilde{f}_\delta, \tilde{g}_\delta)$. Note that the solution \tilde{u} satisfy $\tilde{u}(\cdot, \bar{T}) = 0 = \tilde{u}_t(\cdot, \bar{T})$ in Ω . Now, in order to obtaining the desired control function h we use the trace result available in the previous section. As we have considered the space dimension $N \geq 2$ we can use the Lemma 2.3 to conclude that the trace of the conormal derivative of \tilde{u} is locally square integrable along of the surface Γ , that is $\nu_t \tilde{u}_t - \nabla \tilde{u} \cdot \nu_x \in L^2_{loc}(\Gamma)$. Hence $\nu_t \tilde{u}_t - \nabla \tilde{u} \cdot \nu_x \in L^2(\Gamma_T)$. To finish the prove we define $\hat{u} =: \tilde{u}|_{Q_{\bar{T}}}$ the restriction of \tilde{u} to the domain $Q_{\bar{T}}$ and note that $\hat{u}(\cdot, \bar{T}) = 0 = \hat{u}_t(\cdot, \bar{T})$ in $\Omega_{\bar{T}}$. Now we extend \hat{u} to a function u defined on Q_T by setting $u = 0$ in $Q_T - Q_{\bar{T}}$. After defines $h := \nu_t u_t - \nabla u \cdot \nu_x$ on Γ_T . Note that $u \in H^1(Q_T)$ and satisfy $u(\cdot, T) = 0 = u_t(\cdot, T)$ in Ω_T see that u and h satisfy the conditions of the Theorem 1.2.

4 PROOF OF THE THEOREM 1.1

We begin the proof with a similar construction that one made by Russell in [18]. Let $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ and Ω_δ a δ -neighborhood of the domain Ω . Let $E : H^1(\Omega) \times L^2(\Omega) \rightarrow H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ be a bounded linear operator that takes the pair $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ to the pair $(\tilde{w}_0, \tilde{w}_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, where \tilde{w}_0 and \tilde{w}_1 are extension of w_0 and w_1 to \mathbb{R}^N respectively, both with compact support in Ω_δ . Let $w \in H^1_{loc}(\mathbb{R}^N \times \mathbb{R})$ be the solution of the Cauchy problem

$$\frac{\partial^2 w}{\partial t^2} - \Delta w + c^2 w = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R} \tag{4.1}$$

$$w(\cdot, 0) = \tilde{w}_0, \quad w_t(\cdot, 0) = \tilde{w}_1 \quad \text{in} \quad \mathbb{R}^N. \tag{4.2}$$

Being w the solution of the Cauchy problem (4.1)-(4.2) and taking account the operator $S_T : H^1(\Omega_\delta) \times L^2(\Omega_\delta) \rightarrow H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ as defined in the Section 2. By the Lemma 2.1, with $U = \Omega_\delta$, we have the existence of positive real constants $T_0 > d(\Omega_\delta)$ and K , independent of \tilde{w}_0 and \tilde{w}_1 such that is valid the estimate

$$\|\mathfrak{R}_{\Omega_\delta} S_T(\tilde{w}_0, \tilde{w}_1)\|_{H^1(\Omega_\delta) \times L^2(\Omega_\delta)}^2 \leq \frac{K}{t^N} \|(\tilde{w}_0, \tilde{w}_1)\|_{H^1(\Omega_\delta) \times L^2(\Omega_\delta)}^2 \tag{4.3}$$

for every $t \geq T_0$ and $\mathfrak{R}_{\Omega_\delta}$ is the restriction operator to Ω_δ .

Now, let $\varphi \in C^\infty_0(\mathbb{R}^N)$ be a cut off function such that $\varphi \equiv 1$ in $\Omega_{\frac{\delta}{2}}$ and $\varphi \equiv 0$ in $\mathbb{R}^N \setminus \Omega_{\frac{3\delta}{2}}$. Note that $(\varphi(\cdot)w(\cdot, T), \varphi(\cdot)w_t(\cdot, T)) \in H^1_0(\Omega_\delta) \times L^2(\Omega_\delta)$. Following, we solve the Cauchy problem

$$\frac{\partial^2 z}{\partial t^2} - \Delta z + c^2 z = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R} \tag{4.4}$$

$$z(\cdot, T) = \varphi(\cdot)w(\cdot, T), \quad w_t(\cdot, 0) = \varphi(\cdot)w_t(\cdot, T) \quad \text{in} \quad \mathbb{R}^N, \tag{4.5}$$

and we define the operator $\bar{S}_T : H^1(\Omega_\delta) \times L^2(\Omega_\delta) \rightarrow H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ by $\bar{S}_T(z(\cdot, T), z_t(\cdot, T)) = (z(\cdot, 0), z_t(\cdot, 0))$ being z the solution of the Cauchy problem (4.4)-(4.5). It is important to highlight the relationship between the operators \bar{S}_T and S_T as follows: Let P_i be the projection of $H^1(U) \times L^2(U)$ onto H^{1-i} , $i = 0, 1$. Then

$$P_0 \bar{S}_T(u_0, u_1) = P_0 S_T(u_0, -u_1) \tag{4.6}$$

$$P_1 \bar{S}_T(u_0, u_1) = P_1 S_T(-u_0, u_1). \tag{4.7}$$

So, from relations above we can guarantee that the operators \bar{S}_T have the same properties that the operators S_T . So, for $T > d(U)$ and the validity of the inequality

$$\|\mathfrak{R}_{\Omega_\delta} \bar{S}_T(z(\cdot, T), z_t(\cdot, T))\|_{H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)}^2 \leq \frac{K}{t^N} \|(z(\cdot, T), z_t(\cdot, T))\|_{H^1(\Omega_\delta) \times L^2(\Omega_\delta)}^2 \tag{4.8}$$

for every $t \geq T_0$, being T_0 and K as in the inequality (4.3).

Taken w and z the solutions of the Cauchy problems (4.1)-(4.2) and (4.4)-(4.5) respectively, we define $u := w - z$. Note that the function u satisfy

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + c^2 u = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}. \tag{4.9}$$

Observe that by definition of u we have

$$u(\cdot, T) = 0 = u_t(\cdot, T) \quad \text{in} \quad \Omega. \tag{4.10}$$

Keep in mind (4.9) and (4.10), in order of u to be the solution of the Cauchy problem (1.2)-(1.3) satisfying the condition (1.4) it is necessary only guarantee that $w(\cdot, 0) - z(\cdot, 0)$ and $w_t(\cdot, 0) - z_t(\cdot, 0)$ are extension of f and g respectively from Ω to all \mathbb{R}^N . If this is possible then $\tilde{f}_\delta = w(\cdot, 0) - z(\cdot, 0)$ and $\tilde{g}_\delta = w_t(\cdot, 0) - z_t(\cdot, 0)$ be the desirable extension of f and g respectively. That is equivalent to guarantee the solution the existence of solution of the equations

$$w(\cdot, 0) - z(\cdot, 0) = f, \quad w_t(\cdot, 0) - z_t(\cdot, 0) = g \quad \text{in} \quad \Omega.$$

In terms of the operators S_T and \bar{S}_T the equations above are equivalents to the equation

$$E(w_0, w_1) - (\bar{S}_T \varphi S_T E)(w_0, w_1) = (f, g) \quad \text{in} \quad H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N). \tag{4.11}$$

Inserting the operator restriction to Ω , denoted by \mathfrak{R} , the equation (4.11) becomes

$$(I - K_T)(w_0, w_1) = (f, g), \tag{4.12}$$

being $K_T = \mathfrak{R} \bar{S}_T \varphi S_T E$ and (w_0, w_1) is the variable of such equation taken on space $H^1(\Omega) \times L^2(\Omega)$. For any $T > d(\Omega)$, K_T is a linear compact operator on the space $H^1(\Omega) \times L^2(\Omega)$. Now, to show that equation (4.12) has solution is equivalent to show the existence of inverse operator $(I - K_T)^{-1}$, for this purpose it is sufficient to guarantee the contractivity of the family of operators

$\{K_T : T > d(\Omega)\}$. It is in this point where the inequalities (4.3) and (4.8) will take place. For $T > T_0$ follows that

$$\begin{aligned} \|K_T(w_0, w_1)\|_{H^1(\Omega) \times L^2(\Omega)}^2 &= \|\Re \bar{S}_T \varphi S_T E(w_0, w_1)\|_{H^1(\Omega) \times L^2(\Omega)}^2 \\ &\leq \frac{K}{t^N} \|\varphi S_T E(w_0, w_1)\|_{H^1(\Omega) \times L^2(\Omega)}^2 \\ &\leq \frac{K^2}{t^N} \|E(w_0, w_1)\|_{H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)}^2 \\ &\leq \frac{CK^2}{t^{2N}} \|(w_0, w_1)\|_{H^1(\Omega) \times L^2(\Omega)}^2, \end{aligned}$$

where K and C are constants independent from w_0 and w_1 . So, from the sequences of inequalities above we obtain, for $T > T_0$

$$\|K_T(w_0, w_1)\|_{H^1(\Omega) \times L^2(\Omega)} \leq \frac{\bar{C}}{t^N} \|(w_0, w_1)\|_{H^1(\Omega) \times L^2(\Omega)}, \tag{4.13}$$

where \bar{C} is a constant dependent only K and C . Thus, by inequality (4.13) we guarantee that K_T is contraction for a $T > T_0$ great sufficiently. This ensures the existence of $T > d(\Omega)$ great sufficiently for which the equation (4.12) has solution. However, we would like to establish a lower bound for values of T for which (4.12) is invertible. It is in this point that analytic extension expressed in the Lemma 2.2 will be useful. Because from relationships (4.6) and (4.7) the family of operators $\{\bar{S}_T : T > d(\Omega)\}$ have the same properties of the family of operators $\{S_T : T > d(\Omega)\}$, so the analytic extension expressed in the Lemma 2.2 is also valid for the family of operators $\{\bar{S}_T : T > d(\Omega)\}$. Thus, we have the guarantee that the family of linear compact operators $\{K_T : T > d(\Omega)\}$ can be extended analytically to a family of linear compact operators $\{K_\zeta : \zeta \in \Sigma_0\}$, where Σ_0 is complex sector $\{\zeta = T_1 + z, |arg(z)| \leq \frac{\pi}{4}\}$, being T_1 any constant greater than $d(\Omega)$. After, we utilize the theorem of alternative of F. V. Atkinson (see [11] p. 370) to the effect either 1 is eigenvalue of each of the operators $K_\zeta, \zeta \in \Sigma_0$ or else $(I - K_\zeta)^{-1}$ exists for all except at most for a finite number of values of ζ in each compact subset of Σ_0 . As observed, for a real $\zeta = T$ sufficiently large K_ζ is a contraction, i.e., 1 is not eigenvalue of K_ζ , hence the later possibility must be the case. So, for all $T \geq T_1$ with the possible exception of a finite number of values, $(I - K_T)^{-1}$ exists, concluding the proof of the Theorem 1.1.

Remark 1. Let $\bar{x} \in \mathbb{R}^N$ be the midpoint of a internal segment of Ω of length $d(\Omega)$. For $r > 0$ let $B(\bar{x}, r)$ denotes the open ball with center in \bar{x} and radius r . Considering the infinite cone

$$C\left(\frac{d(\Omega)}{2} + \delta\right) = \bigcup_{t \geq \frac{d(\Omega)}{2} + \delta} B\left(\bar{x}, t - \frac{d(\Omega)}{2} - \delta\right) \times \{t\}.$$

We suspect that there may be a extension $(\tilde{f}, \tilde{g}) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ of $(f, g) \in H^1(\Omega) \times L^2(\Omega)$ such that the solution of (1.2)-(1.3) with Cauchy data (\tilde{f}, \tilde{g}) vanishes in the finite cone

$\bigcup_{\frac{d(\Omega)}{2} + \delta \leq t \leq T} B\left(\bar{x}, t - \frac{d(\Omega)}{2} - \delta\right) \times \{t\}$, of every $T \geq \frac{d(\Omega)}{2} + \delta$. The confirmation of this fact allow us to obtain control for the linear Klein-Gordon equation in some non-cylindrical domains, with control time for every $T > \frac{d(\Omega)}{2}$ satisfying $\Omega_T \subset B(\bar{x}, T - \frac{d(\Omega)}{2})$. Similar result was obtained in [12] for even dimensional wave equation for $\Omega = B(0, 1)$. We intend to soon return to this subject.

RESUMO. O objetivo deste artigo é estudar um problema de controlabilidade exata na fronteira em domínios não cilíndricos para a equação linear de Klein-Gordon. Aqui, trabalhamos próximo das técnicas de extensão apresentadas por J. Lagnese, em [12], que é baseada no método de controlabilidade de Russell. O tempo de controle é obtido em qualquer instante maior que o valor do diâmetro do domínio no qual os dados iniciais estão suportados. O controle é de quadrado integrável e atua em toda fronteira e é obtido por meio da derivada conormal associada ao operador linear de Klein-Gordon.

Palavras-chave: controle exato na fronteira, domínios não-cilíndricos, equação linear Klein-Gordon.

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