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Integral Representations of Mittag-Leffler Function on the Positive Real Axis

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ABSTRACT. We use the method for finding inverse Laplace transform without using integration on the complex plane to show that the three-parameter Mittag-Leffler function, which appear in many problems associated with fractional calculus, has similar integral representations on the positive real axis. Some of them are presented.

Keywords: inverse Laplace transform, Mittag-Leffler functions, integral representations, fractional calculus.

INTRODUCTION

The Mittag-Leffler function, introduced in 1902 by Gösta Mittag-Leffler [23], is important in many fields, including description of the anomalous dielectric properties, probability theory, statistics, viscoelasticity, random walks and dynamical systems [9, 10, 11, 14, 19, 25, 26]. Successively, generalizations of Mittag-Leffler function were proposed [27]. These functions play a fundamental role in arbitrary order calculus, popularly known as fractional calculus [4, 13, 18, 20, 22, 29], as well as the exponential function play in integer order calculus.

The classical Laplace transform is one of the most widely tools used in the literature for solving integral equations and ordinary or partial differential equations, involving integer or fractional order derivatives [1, 8, 31, 33]. It is also used in many others applications such as electrical circuit and signal processing [15, 21, 35]. In general, the Laplace inversion is done numerically due to the impossibility of the exact inversion by means of an integration on the complex plane [7, 32].

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M. N. Berberan-Santos [2] proposed a new methodology for evaluation of the numerical inverse Laplace transform, without using integration on the complex plane, which was published in 2005, and its methodology was used recently, for instance, to discuss the luminescence decay of inorganic solids, and to obtain an integral representation of Mittag-Leffler relaxation function, a special one-parameter Mittag-Leffler function [3]. Recently, the method to evaluate the inverse Laplace transform without using integration on the complex plane was applied in [6] to find integral representations on the positive real axis for some functions.

In this paper, with the method for finding inverse Laplace transform without using integration on the complex plane we show that the three-parameter Mittag-Leffler function has integral representations on the positive real axis.

The paper is organized as follows: in Section 1, we present some preliminaries concepts and the methodology of inversion of the Laplace transform. In Section 2, using this methodology, we express the integral representations of three-parameter Mittag-Leffler function and we use the results from this study to discuss, in Section 3, a class of improper integrals, expressing them in terms of the Mittag-Leffler functions. Concluding remarks close the paper.

1 PRELIMINARIES

In this section, we present the definition and some special cases of the Mittag-Leffler functions, and a review of the methodology of inversion of the Laplace transform proposed by M. N. Berberan-Santos in the following subsections.

1.1 Mittag-Leffler functions

The three-parameter Mittag-Leffler function, introduced by Prabhakar [27], of complex variable $z \in \mathbb{C}$, with complex parameters $\alpha, \beta, \gamma \in \mathbb{C}$, is defined by[1]

$$\mathrm{E}^{\gamma}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j z^j}{\Gamma(\alpha j+\beta)\, j!}, \tag{1.1}$$

with $\mathscr{R}(\alpha) > 0, \mathscr{R}(\beta) > 0$ and $\mathscr{R}(\gamma) > 0$, where

$$\Gamma(\rho) = \int_0^{\infty} t^{\rho-1} e^{-t}\mathrm{d}t, \tag{1.2}$$

is the Gamma function, and

$$(\gamma)_j := \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)}, \tag{1.3}$$

is the Pochhammer symbol. Taking $\gamma = 1$ in equation (1.1), we get the two-parameter Mittag-Leffler function:

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)}. \tag{1.4}$$

[1] $\mathscr{R}[\xi]$ indicates the real part of ξ.

When $\beta = 1$ in equation (1.4), we get the standard Mittag-Leffler function [23, 34]:

$$\mathrm{E}_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}. \tag{1.5}$$

A function $f \in \mathrm{C}^{\infty}(\mathrm{I})$ is to be completely monotonic (CM) on interval I if $(-1)^k f^k(x) \geq 0$ for $x \in \mathrm{I}$ and $k \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ [33]. Capelas et al. [5] showed that the function $\varphi(t) = t^{\beta-1}\mathrm{E}^{\gamma}_{\alpha,\beta}(-t^{\alpha})$ is CM for $0 < \alpha\gamma \leq \beta \leq 1$. Particularly, these functions play an important rule in anomalous dielectric relaxation where the memory effect appears specifically in the Havriliak-Negami model, which contains Davidson-Cole model, Cole-Cole model and the classical Debye model, as particular cases [17, 24].

An interesting functional relation case and the more simple special relations involving the Mittag-Leffler functions $\mathrm{E}_{\alpha,\beta}(z)$ and $\mathrm{E}_{\alpha}(z)$, with $z \in \mathbb{C}$, are given by the following equations:

$$\mathrm{E}_{2\alpha}(z^2) = \frac{1}{2}\left[\mathrm{E}_{\alpha}(z) + \mathrm{E}_{\alpha}(-z)\right]. \tag{1.6}$$

$$\mathrm{E}_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}. \tag{1.7}$$

$$\mathrm{E}_{1,2}(z) = \frac{e^z - 1}{z}. \tag{1.8}$$

$$\mathrm{E}_{2}(-z^2) = \cos z. \tag{1.9}$$

$$\mathrm{E}_{2}(z^2) = \cosh z. \tag{1.10}$$

1.2 Inversion of the Laplace transform

Let $f(t)$ be a real function of (time) variable $t \geq 0$. The Laplace transform of f, denoted by $\mathrm{F}(s) = \mathscr{L}[f](s)$, is defined as follows:

$$\mathscr{L}[f](s) = \mathrm{F}(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t, \tag{1.11}$$

whenever the integral converges for[2] $\mathscr{R}[s] \geq \sigma > 0$, where $s = \sigma + i\tau$, with σ and τ real numbers, and $\mathrm{F}(s) = 0$ for $\sigma < 0$. By means of equation (1.11) with Euler formula, we observe that

$$\mathscr{R}[\mathrm{F}(\sigma + i\tau)] = \int_0^{\infty} e^{-\sigma t} f(t)\cos(t\tau)\mathrm{d}t \tag{1.12}$$

and

$$\mathscr{I}[\mathrm{F}(\sigma + i\tau)] = -\int_0^{\infty} e^{-\sigma t} f(t)\sin(t\tau)\mathrm{d}t. \tag{1.13}$$

The expression for evaluation of the inverse Laplace transform of $\mathrm{F}(s)$, proposed by M. N. Berberan-Santos [2], is given by

[2] $\mathscr{R}[s]$ indicates the real part of s and the imaginary part is denoted by $\mathscr{I}[s]$.

$$f(t) = \frac{e^{\sigma t}}{\pi} \int_0^\infty \left[\mathscr{R}\left[\mathrm{F}(\sigma + i\tau)\right]\cos(t\tau) - \mathscr{I}\left[\mathrm{F}(\sigma + i\tau)\right]\sin(t\tau)\right]\mathrm{d}\tau, \tag{1.14}$$

for $t > 0$ and any real number σ satisfying the condition $\sigma \geq \sigma_0 > 0$, where σ_0 is large enough that $\mathrm{F}(s)$ is defined for $\mathscr{R}[s] \geq \sigma_0 > 0$. The expression in equation (1.14) recovers the real function whose Laplace transform is known.

From equation (1.14), for $t > 0$, we can write

$$\frac{\pi}{2}\left[e^{-\sigma t} f(t) + e^{\sigma t} f(-t)\right] = \int_0^\infty \mathscr{R}\left[\mathrm{F}(\sigma + i\tau)\right] \cos(t\tau)\mathrm{d}\tau. \tag{1.15}$$

The function f is such that $f(\xi) = 0$ for $\xi < 0$. Since $t > 0$, equation (1.15) yields

$$f(t) = \frac{2e^{\sigma t}}{\pi} \int_0^\infty \mathscr{R}\left[\mathrm{F}(\sigma + i\tau)\right]\cos(t\tau)\mathrm{d}\tau, \tag{1.16}$$

Furthermore, in a similar way,

$$f(t) = -\frac{2e^{\sigma t}}{\pi} \int_0^\infty \mathscr{I}\left[\mathrm{F}(\sigma + i\tau)\right]\sin(t\tau)\mathrm{d}\tau. \tag{1.17}$$

Namely, there are three possible cases to find the inverse Laplace transform of a function $\mathrm{F}(s)$, they are given by equations (1.14), (1.16) and (1.17).

In this point, it is important to consider a simple example, illustrating the methodology that will be used in this work: The Laplace transform of the exponential function is given by $\mathrm{F}(s) = \frac{1}{s-1}$, for $\mathscr{R}[s] > 0$. Choosing $\mathscr{R}[s] = \sigma = 2$ and writing $s = 2 + i\tau$, we have that $\mathscr{R}[\mathrm{F}(2 + i\tau)] = \frac{1}{1+\tau^2}$. Thus, from equation (1.16), we obtain

$$\frac{\pi}{2} e^{-t} = \int_0^\infty \frac{\cos(t\tau)}{1+\tau^2}\mathrm{d}\tau, \qquad \text{for } \ t > 0. \tag{1.18}$$

2 INTEGRAL REPRESENTATIONS OF MITTAG-LEFFLER FUNCTION

Some integral representations associated with the one-parameter Mittag-Leffler function can be found in the following papers: [3, 12, 16]. Here we present integral representations for the three-parameter Mittag-Leffler function and to prove the representations, we use the relations in equations (1.14), (1.16) and (1.17). It is worthwhile to mention that the detail treatment of the similar study can be found in [28, 30].

Theorem 1. Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\lambda \in \mathbb{R}$. Then, for $t > 0$, the three-parameter Mittag-Leffler function $\Psi(t) = \mathrm{E}^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})$ has the following integral representations on the positive real axis

$$\begin{aligned}
\mathrm{E}_{\alpha,\beta}^{\gamma}(\lambda\, t^{\alpha}) &= \frac{t^{1-\beta}\, e^{\sigma t}}{\pi} \int_0^\infty \frac{r^{\alpha\gamma-\beta}}{\tilde{r}} \cos\left[\theta\,(\alpha\gamma-\beta)-\tilde{\theta}+t\tau\right] \mathrm{d}\tau, &(2.1)\\
&= \frac{2t^{1-\beta}\, e^{\sigma t}}{\pi} \int_0^\infty \frac{r^{\alpha\gamma-\beta}}{\tilde{r}} \cos\left[\theta\,(\alpha\gamma-\beta)-\tilde{\theta}\right] \cos(t\tau)\mathrm{d}\tau, &(2.2)\\
&= -\frac{2t^{1-\beta}\, e^{\sigma t}}{\pi} \int_0^\infty \frac{r^{\alpha\gamma-\beta}}{\tilde{r}} \sin\left[\theta\,(\alpha\gamma-\beta)-\tilde{\theta}\right] \sin(t\tau)\mathrm{d}\tau, &(2.3)
\end{aligned}$$

where $\sigma > \sigma_0$ and σ_0, r, θ, $\tilde{\theta}$ and $\tilde{r}$ are defined by equations:

$$\sigma_0 = |\lambda|^{\frac{1}{\alpha}}. \tag{2.4}$$

$$r\cos\theta = \sigma \qquad \text{and} \qquad r\sin\theta = \tau. \tag{2.5}$$

$$\tilde{r}^{\frac{1}{\gamma}} \cos\left(\frac{\tilde{\theta}}{\gamma}\right) = r^{\alpha}\cos(\theta\alpha)-\lambda \qquad \text{and} \qquad \tilde{r}^{\frac{1}{\gamma}} \sin\left(\frac{\tilde{\theta}}{\gamma}\right) = r^{\alpha}\sin(\theta\alpha). \tag{2.6}$$

Proof. The Laplace transform of the three-parameter Mittag-Leffler type function $f(t) = t^{\beta-1}\mathrm{E}_{\alpha,\beta}^{\gamma}(\lambda\, t^{\alpha})$ is given by

$$\mathscr{L}\left[t^{\beta-1}\mathrm{E}_{\alpha,\beta}^{\gamma}(\lambda\, t^{\alpha})\right](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\lambda)^{\gamma}} = \mathrm{F}(s), \qquad \text{for } |\lambda s^{-\alpha}| < 1. \tag{2.7}$$

The complex parameter s can be written as

$$s = \sigma + i\,\tau = re^{i\theta}, \tag{2.8}$$

with σ, $\tau \in \mathbb{R}$, $r > 0$ and $0 \le \theta \le 2\pi$. In this way, from equation (2.8), we get equations in (2.5).

Expression $(s^{\alpha}-\lambda)^{\gamma}$ in the denominator of $\mathrm{F}(s)$ can be written in the following form:

$$(s^{\alpha}-\lambda)^{\gamma} = \tilde{r}e^{i\tilde{\theta}}. \tag{2.9}$$

Replacing s by $re^{i\theta}$ in equation (2.9), we get

$$\left(r^{\alpha}e^{i\theta\alpha}-\lambda\right)^{\gamma} = \tilde{r}e^{i\tilde{\theta}},$$

that is,

$$r^{\alpha}e^{i\theta\alpha}-\lambda = \tilde{r}^{\frac{1}{\gamma}}e^{i\frac{\tilde{\theta}}{\gamma}}. \tag{2.10}$$

Separating real part and imaginary part in equation (2.10), we obtain the expressions in equation (2.6).

We can thus conclude that

$$\mathrm{F}(s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\lambda)^{\gamma}} = \frac{\left(r\,e^{i\theta}\right)^{\alpha\gamma-\beta}}{\tilde{r}\,e^{i\tilde{\theta}}} = \frac{r^{\alpha\gamma-\beta}}{\tilde{r}}\, e^{i\left[\theta(\alpha\gamma-\beta)-\tilde{\theta}\right]}.$$

Through manipulation of $\mathrm{F}(s)$, we can separate its real and imaginary parts:

$$\mathscr{R}\left[\mathrm{F}(\sigma+i\tau)\right]=\frac{r^{\alpha\gamma-\beta}}{\tilde{r}}\cos\left[\theta\left(\alpha\gamma-\beta\right)-\tilde{\theta}\right] \tag{2.11}$$

and

$$\mathscr{I}\left[\mathrm{F}(\sigma+i\tau)\right]=\frac{r^{\alpha\gamma-\beta}}{\tilde{r}}\sin\left[\theta\left(\alpha\gamma-\beta\right)-\tilde{\theta}\right]. \tag{2.12}$$

Substituting equations (2.11) and (2.12) into equations (1.14), (1.16) and (1.17), we arrive at equations (2.1), (2.2) and (2.3), respectively; and finally if we choose $\sigma>\sigma_0=|\lambda|^{\frac{1}{\alpha}}$, then the inequality $|\lambda\, s^{-\alpha}|<1$ is satisfied. ☐

According to Theorem 1, the Mittag-Leffler function has similar integral representations, as we have seen in the equations (2.1)-(2.3). We present some applications of this theorem in the next section.

3 EVALUATION OF A CLASS OF IMPROPER INTEGRALS

In what follows we will discuss some evaluations for improper integrals using Theorem 1 for specific values of the parameters appearing in equations (2.1)-(2.3). As by-products, in the following examples, interesting integrals are obtained.

We should point out that we consider the case $|\lambda|=1$ in the next illustrative examples. In this way, from equation (2.4), we can choose $\sigma=2>\sigma_0=1^{\frac{1}{\alpha}}=1$. Equation (2.5) with $\sigma=2$ imply that

$$r\cos\theta=2 \qquad \text{and} \qquad r\sin\theta=\tau. \tag{3.1}$$

By equation (3.1), we have

$$r=\sqrt{4+\tau^2} \qquad \text{and} \qquad \theta=\arccos\left(\frac{2}{\sqrt{4+\tau^2}}\right)=\arcsin\left(\frac{\tau}{\sqrt{4+\tau^2}}\right). \tag{3.2}$$

Example 1. We consider the function:

$$\mathrm{E}_{1,\beta}^{\gamma}(-t)=\frac{1}{\Gamma(\beta)}\,{}_1\mathrm{F}_1\left(\gamma;\beta;-t\right),$$

where ${}_1\mathrm{F}_1\left(\gamma;\beta;t\right)$ be a confluent hypergeometric function [18]. In particular, if $0<\gamma\leq\beta\leq 1$, the function $\varphi(t)=t^{\beta-1}\mathrm{E}_{1,\beta}^{\gamma}(-t)$ is CM.

From equation (2.1), we can derive that

$$\mathrm{E}_{1,\beta}^{\gamma}(-t)=\frac{2t^{1-\beta}\,e^{2t}}{\pi}\int_0^{\infty}\frac{r^{\gamma-\beta}}{\tilde{r}}\cos\left[\theta\left(\gamma-\beta\right)-\tilde{\theta}\right]\cos\left(t\tau\right)\mathrm{d}\tau, \tag{3.3}$$

or in a different form, we can obtain an integral representation for confluent hypergeometric function as follows:

$${}_1\mathrm{F}_1\left(\gamma;\beta;-t\right)=\frac{2\,\Gamma(\beta)\,t^{1-\beta}\,e^{2t}}{\pi}\int_0^{\infty}\frac{r^{\gamma-\beta}}{\tilde{r}}\cos\left[\theta\left(\gamma-\beta\right)-\tilde{\theta}\right]\cos\left(t\tau\right)\mathrm{d}\tau, \tag{3.4}$$

where, according to equation (2.6),

$$\tilde{\theta} = \gamma \arctan\left(\frac{\tau}{4}\right) \qquad \text{and} \qquad \tilde{r} = \left(4+\tau^2\right)^{\gamma/2}. \tag{3.5}$$

Substituting equations (3.2) and (3.5) into equation (3.4), we obtain the result

$${}_1\mathrm{F}_1\left(\gamma;\beta;-t\right) = \frac{2\Gamma(\beta)\,t^{1-\beta}\,e^{2t}}{\pi}\int_0^\infty \phi\left(t,\tau\right)\cos\left(t\tau\right)\mathrm{d}\tau, \tag{3.6}$$

for $t>0$, where

$$\phi\left(t,\tau\right) = \frac{1}{\left(4+\tau^2\right)^{\beta/2}}\cos\left[\left(\gamma-\beta\right)\arccos\left(\frac{2}{\sqrt{4+\tau^2}}\right) - \gamma\arctan\left(\frac{\tau}{4}\right)\right]. \tag{3.7}$$

Taking $\gamma=\beta$ in equation (3.7), we have

$${}_1\mathrm{F}_1\left(\beta;\beta;-t\right) = \frac{2\Gamma(\beta)\,t^{1-\beta}\,e^{2t}}{\pi}\int_0^\infty \frac{\cos\left[\beta\,\arctan\left(\frac{\tau}{4}\right)\right]\,\cos\left(t\tau\right)}{\left(4+\tau^2\right)^{\beta/2}}\,\mathrm{d}\tau. \tag{3.8}$$

According to equation (3.8) with $\mathrm{E}^{\beta}_{1,\beta}(-t) = \dfrac{1}{\Gamma(\beta)}\,{}_1\mathrm{F}_1\left(\beta;\beta;-t\right)$ we thus have

$$\mathrm{E}^{\beta}_{1,\beta}(-t) = \frac{2t^{1-\beta}\,e^{2t}}{\pi}\int_0^\infty \frac{\cos\left[\beta\,\arctan\left(\frac{\tau}{4}\right)\right]\,\cos\left(t\tau\right)}{\left(4+\tau^2\right)^{\frac{\beta}{2}}}\,\mathrm{d}\tau, \tag{3.9}$$

for $t>0$.

If $\gamma=1$ and $\beta=2$, by equations (3.3) and (1.8),

$$\mathrm{E}_{1,2}(-t) = \frac{1-e^{-t}}{t} = \frac{2e^{2t}}{t\,\pi}\int_0^\infty \frac{\cos(\theta+\tilde{\theta})\,\cos\left(t\tau\right)}{r\,\tilde{r}}\,\mathrm{d}\tau, \tag{3.10}$$

that is,

$$\frac{\pi}{2}\,e^{-2t}\left(1-e^{-t}\right) = \int_0^\infty \frac{1}{r^2\,\tilde{r}^2}\left(r\cos\theta\,\tilde{r}\cos\tilde{\theta} - r\sin\theta\,\tilde{r}\sin\tilde{\theta}\right)\cos\left(t\tau\right)\mathrm{d}\tau. \tag{3.11}$$

Equations (2.6) and (3.1) provided that

$$\tilde{r}\cos\tilde{\theta} = 3 \qquad \text{and} \qquad \tilde{r}\sin\tilde{\theta} = \tau. \tag{3.12}$$

Taking into account the equations (3.12) and (3.1), we can rewrite equation (3.11) in the respective form:

$$\frac{\pi}{2}\,e^{-2t}\left(1-e^{-t}\right) = \int_0^\infty \frac{\left(6-\tau^2\right)\cos\left(t\tau\right)}{\tau^4+13\tau^2+36}\,\mathrm{d}\tau, \qquad \text{for } t>0. \tag{3.13}$$

Example 2. In this example we consider the function: $\mathrm{E}_{\alpha,\alpha}\left(-t^{\alpha}\right)$. In particular, if $0<\alpha\leq 1$, the function $\varphi(t) = t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(-t^{\alpha}\right)$ is CM.

In this case, if we use equation (2.3), since $\alpha\gamma-\beta=0$, then we have

$$\mathrm{E}_{\alpha,\alpha}(-t^{\alpha})=\frac{2t^{1-\alpha}e^{2t}}{\pi}\int_0^{\infty}\frac{\sin\tilde{\theta}\,\sin(t\tau)}{\tilde{r}}\,\mathrm{d}\tau, \tag{3.14}$$

where $t>0$ and $\tilde{\theta}$ and $\tilde{r}$ are defined in equation (2.6), given by

$$\tilde{r}=\sqrt{r^{2\alpha}+2r^{\alpha}\cos(\theta\alpha)+1}\qquad\text{and}\qquad \sin\tilde{\theta}=\frac{r^{\alpha}\sin(\theta\alpha)}{\tilde{r}}, \tag{3.15}$$

where r and θ are given by equation (3.1).

When $\alpha=2$, then equation (2.6), in accordance with equation (3.1), yields the following formula

$$\tilde{r}\cos\tilde{\theta}=r^2\cos(2\theta)+1=5-\tau^2\qquad\text{and}\qquad \tilde{r}\sin\tilde{\theta}=r^2\sin(2\theta)=4\tau. \tag{3.16}$$

Multiplying the integrand in equation (3.14) by $\frac{\tilde{r}}{\tilde{r}}$, and substituting equations (3.1) and (3.16) into (3.14), we thus derive the following integral representation

$$\mathrm{E}_{2,2}(-t^2)=\frac{2e^{2t}}{t\pi}\int_0^{\infty}\frac{4\tau\sin(t\tau)}{\tau^4+6\tau^2+25}\,\mathrm{d}\tau,\qquad \text{for } t>0. \tag{3.17}$$

Equation (1.7) imply the following result

$$\mathrm{E}_{2,2}\left(-t^2\right)=\frac{\sinh it}{it}=\frac{\sin t}{t}.$$

Then, equation (3.17) can be rewritten in the alternative form:

$$e^{-2t}\sin t=\frac{2}{\pi}\int_0^{\infty}\frac{4\tau\sin(t\tau)}{\tau^4+6\tau^2+25}\,\mathrm{d}\tau,\qquad \text{for } t>0. \tag{3.18}$$

Example 3. In the last case we consider the function: $\mathrm{E}_{\alpha}\left(-t^{\alpha}\right)$. In particular, if $0<\alpha\leq 1$, the function $\varphi(t)=\mathrm{E}_{\alpha}\left(-t^{\alpha}\right)$ is CM. From equation (2.2), we obtain the following integral representation

$$\mathrm{E}_{\alpha}(-t^{\alpha})=\frac{2e^{2t}}{\pi}\int_0^{\infty}\frac{r^{\alpha-1}}{\tilde{r}}\cos\left[\theta\left(\alpha-1\right)-\tilde{\theta}\right]\cos\left(t\tau\right)\mathrm{d}\tau, \tag{3.19}$$

for $t>0$, where r and θ are given by equation (3.1) and $\tilde{r}$ and $\tilde{\theta}$ are given by equation (2.6).

Moreover, when manipulating the mathematical expression in equation (3.19), we can give another similar integral representation as follows

$$\mathrm{E}_{\alpha}(-t^{\alpha})=\frac{2e^{2t}}{\pi}\int_0^{\infty}\frac{r^{\alpha-2}\left[2r^{\alpha}+2\cos\left(\theta\alpha\right)+\tau\sin\left(\theta\alpha\right)\right]\cos\left(t\tau\right)}{r^{2\alpha}+2r^{\alpha}\cos\left(\theta\alpha\right)+1}\,\mathrm{d}\tau. \tag{3.20}$$

In particular, when $\alpha=1$ in equation (3.20), we obtain another integral representation for the exponential function:

$$\frac{\pi}{2}e^{-t}=\int_0^{\infty}\frac{3\cos\left(\frac{t\tau}{3}\right)}{9+\tau^2}\,\mathrm{d}\tau,\qquad \text{for } t>0. \tag{3.21}$$

When $\alpha = 2$, by using equation (1.9) and the integral representation type in equation (2.3), the cosine function takes the form:

$$\cos t = \mathrm{E}_2(-t^2) = \frac{2\,e^{2t}}{\pi}\int_0^\infty \frac{r}{\tilde{r}}\,\sin(\tilde{\theta}-\theta)\,\sin(t\tau)\mathrm{d}\tau. \tag{3.22}$$

The integral in equation (3.22) can be simplified to

$$\cos t = \frac{2\,e^{2t}}{\pi}\int_0^\infty \frac{\left(\tau^3+3\tau\right)\,\sin(t\tau)}{\tau^4+6\tau^2+25}\,\mathrm{d}\tau, \qquad \text{for } \; t>0. \tag{3.23}$$

Furthermore, using the relation in equation (1.10) and the equation (2.3), the hyperbolic cosine function can be represented by

$$\cosh t = \frac{2\,e^{2t}}{\pi}\int_0^\infty \frac{\left(\tau^3+5\tau\right)\,\sin(t\tau)}{\tau^4+10\tau^2+9}\,\mathrm{d}\tau, \qquad \text{for } \; t>0. \tag{3.24}$$

Finally, we can use the relation in equation (1.6) and the above results to express the function

$$\mathrm{E}_4\left(t^4\right) = \frac{1}{2}\left[\mathrm{E}_2\left(t^2\right)+\mathrm{E}_2\left(-t^2\right)\right] = \frac{\cos t+\cosh t}{2}.$$

In fact, equations (3.23) and (3.24) provided that

$$\mathrm{E}_4\left(t^4\right) = \frac{e^{2t}}{\pi}\int_0^\infty \left[\frac{\left(\tau^3+3\tau\right)}{\tau^4+6\tau^2+25}+\frac{\left(\tau^3+5\tau\right)}{\tau^4+10\tau^2+9}\right]\sin(t\tau)\,\mathrm{d}\tau, \tag{3.25}$$

for $t>0$.

4 CONCLUDING REMARKS

We build similar integral representations for the three-parameter Mittag-Leffler function on the positive real axis using the method for finding inverse Laplace transform without using integration on the complex plane. Many authors have demonstrated interest in the study of the asymptotic behavior of the Mittag-Leffler functions on the interpretation of the solutions of problems associated with fractional diffusion. In this way the integral representations presented in this paper can be used to analyze the asymptotic behavior of these functions. Furthermore, this representation can express improper integrals in terms of trigonometric functions by means of the Mittag-Leffler functions and the presented examples complement corresponding integral representations.

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RESUMO. Através do método para encontrar a transformada de Laplace inversa sem o uso de um contorno de integração no plano complexo, mostramos que a função de Mittag-Leffler de três parâmetros, que aparece em muitos problemas associados com cálculo fracionário, possui representações integrais similares no semieixo real positivo. Algumas delas são apresentadas.

Palavras-chave: transformada de Laplace inversa, funções de Mittag-Leffler, representações integrais, cálculo fracionário.

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