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Locating Eigenvalues of Perturbed Laplacian Matrices of Trees†

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ABSTRACT. We give a linear time algorithm to compute the number of eigenvalues of any perturbed Laplacian matrix of a tree in a given real interval. The algorithm can be applied to weighted or unweighted trees. Using our method we characterize the trees that have up to 5 distinct eigenvalues with respect to a family of perturbed Laplacian matrices that includes the adjacency and normalized Laplacian matrices as special cases, among others.

Keywords: perturbed Laplacian matrix, eigenvalue location, trees.

1 INTRODUCTION

The *Spectral Graph Theory* studies the relations between the spectrum of matrices associated to graphs and structural properties of the graphs. The most commonly used representation matrix of a graph is the *adjacency matrix*. If G is a simple undirected graph with vertices $v_1, v_2, \ldots, v_n$, the adjacency matrix $A = (a_{ij})$ of G is the real symmetric matrix of order n with entries 0 or 1, where $a_{ij} = 1$ if and only if vertices v_i and v_j are adjacent.

A *tree* is a connected graph with no cycles. In 2011, Jacobs & Trevisan [7] gave a linear time algorithm to compute the number of eigenvalues of a tree in a given real interval. Their method has the important advantage of being executed directly on the tree so that the matrix is not needed explicitly. The authors observed that the algorithm had potential to be adapted to other matrices, for instance the *Laplacian matrix*, defined as the matrix $L = D_G - A$, where D_G is the diagonal matrix whose diagonal entry ii is the degree d_i of vertex v_i of a graph G and A is the adjacency matrix of G.

In fact, the algorithm of Jacobs and Trevisan and the extensions that followed it became a practical and efficient tool in Spectral Graph Theory. We notice in particular the work of Fritscher

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et al. [5], where the original algorithm was adapted for the Laplacian matrix of a tree and applied to prove that among all trees with n vertices, the star S_n has the highest *Laplacian energy*, which was conjectured by Radenković & Gutman in [8]. Besides, Braga et al. [3] adapted the original algorithm for the *normalized Laplacian matrix*, introduced by Chung [4] as the matrix $\mathcal{L} = (\ell_{ij})$, where $\ell_{ij} = 1$ if $i = j$ and $d_i > 0$, $\ell_{ij} = -\frac{1}{\sqrt{d_i \cdot d_j}}$, if vertices v_i and v_j are adjacent, and $\ell_{ij} = 0$, otherwise. This variation of the algorithm was used to study the multiplicity of normalized Laplacian eigenvalues of small diameter trees, which allowed the authors to characterize the trees that have up to 5 distinct normalized Laplacian eigenvalues.

The results obtained with the localization algorithm for different representation matrices of trees motivated the development of an algorithm to localize the eigenvalues of a tree for a more general class of matrices, generalizing the previous algorithms, which is the aim of this work.

A *weighted graph* is a graph where a real number $\omega(e_{ij}) = \omega_{ij}$ is assigned to each edge e_{ij} connecting vertices v_i and v_j. We say that ω_{ij} is the *weight* of edge e_{ij}. An unweighted graph can be considered as a graph where all edges have weight 1. In [1], Bapat et al. defined the *perturbed Laplacian matrix* of a graph with positive weights, which encompasses the adjacency, Laplacian and normalized Laplacian matrices, among others. Given a real diagonal matrix D, the perturbed Laplacian matrix of G with respect to D, is the matrix

$$\overset{D}{L}(G) = D - A,$$

where $A = (a_{ij})$ is the adjacency matrix of G, with $a_{ij} = \omega_{ij}$ if vertices v_i and v_j are adjacent, and 0 otherwise.

The general idea of the localization algorithm for a perturbed Laplacian matrix of a weighted tree, called *DiagonalizeW*, is the same of the previous algorithms: performing computations directly on the tree, obtain a diagonal matrix D_α congruent to $M + \alpha I$, where M is a representation matrix of a tree.

Beyond preserving the practicality of the original algorithm and its extensions, our method has the advantages of considering weighted trees and allowing to simultaneously derive results for several representation matrices. In fact, the previous localization algorithms are special cases of algorithm *DiagonalizeW*. For an unweighted tree, if D is the zero matrix then $\overset{D}{L}(G) = -A$ and *DiagonalizeW* coincides with the algorithm given in [7]. If D is the diagonal matrix of the degrees of the vertices of G, then $\overset{D}{L}(G)$ is the Laplacian matrix of G and *DiagonalizeW* coincides with the algorithm applied in [5]. Besides, if D is the identity matrix and we take $\omega_{ij} = \frac{1}{\sqrt{d_i \cdot d_j}}$, *DiagonalizeW* is the algorithm for the normalized Laplacian matrix given in [3].

In Section 2 we present algorithm *DiagonalizeW*. In Section 3 we apply this method to characterize the trees that have up to 5 distinct eigenvalues with respect to a family of perturbed Laplacian matrices that includes the adjacency and normalized Laplacian matrices as special cases, among others.

2 LOCATING EIGENVALUES OF PERTURBED LAPLACIAN MATRICES

Taking as input a weighted tree T of order n, a scalar $\alpha \in \mathbb{R}$ and a real diagonal matrix D, the algorithm *DiagonalizeW*(T, α) that we present in this section diagonalizes the matrix $\mathcal{L}^D(T)+\alpha I$, where $\mathcal{L}^D(T)$ is the perturbed Laplacian matrix of T with respect to D. The output is a diagonal matrix D_α *congruent* to $\mathcal{L}^D(T)+\alpha I$. We recall that two square matrices of order n, A and B, are congruent if there is an invertible matrix P such that $A = P^T B P$.

Like the original algorithm of Jacobs and Trevisan for the adjacency matrix, our method is executed directly on the tree T, so that the matrix is not needed explicitly.

The tree T is rooted at an arbitrary vertex and the vertices are ordered $v_1, \ldots, v_n$, so that if v_j is a child of v_k, then $j > k$. Thus the root is the vertex v_1. Every vertex v of T, except for the root, has a *parent*, which is the vertex adjacent to v that is not a child of v.

During the execution the algorithm *DiagonalizeW*(T, α) assigns, to each vertex v_i of T, a real value $a(v_i)$ which at the end corresponds precisely to the entry ii of the diagonal matrix D_α. We call $a(v_i)$ the *diagonal value* of vertex v_i. Initially, each v_i receives the diagonal value $a(v_i) = \delta_i + \alpha$, where δ_i is the entry ii of the diagonal matrix D. Then the vertices are processed bottom-up, towards the root, as described below. For a vertex v_k, we denote by C_k the set of all children of v_k. If v_k is a leaf which is not the root, then $C_k = \emptyset$.

Algorithm 1 - *DiagonalizeW*(T, α).

Input: weighted tree T with ordered vertices $v_1, v_2, \ldots, v_n$, scalar α, diagonal matrix D.
Output: diagonal matrix D_α congruent to $\mathcal{L}^D(T)+\alpha I$.

Initialize $a(v_i) := \delta_i + \alpha$, for each vertex v_i.
For $k = n$ to 1
  if v_k is not a leaf then
    1. if $a(v_i) \neq 0$, for all $v_i \in C_k$, then

$$a(v_k) \leftarrow a(v_k) - \sum_{v_i \in C_k} \frac{(\omega_{ik})^2}{a(v_i)}.$$

    2. if $a(v_i) = 0$ for some $v_i \in C_k$, then
      select one vertex v_j in C_k for which $a(v_j) = 0$;

$$a(v_k) \leftarrow -\frac{(\omega_{jk})^2}{2}; \quad a(v_j) \leftarrow 2;$$

      if v_k has a parent v_ℓ, remove the edge $v_k v_\ell$.
Print $a(v_1), a(v_2), \ldots, a(v_n)$.

To understand how the procedure above computes the diagonal values of a diagonal matrix congruent to the matrix $L(T)+\alpha I$, let us consider a vertex v_k of T with a child v_j, which corresponds to the entries in the matrix below:

$$\begin{array}{c} \\ k \\ \\ j \\ \\ \end{array}\begin{bmatrix} & \vdots & & \vdots & \\ \cdots & a(v_k) & \ldots & \omega_{kj} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & \omega_{jk} & \cdots & a(v_j) & \cdots \\ & \vdots & & \vdots & \end{bmatrix}.$$

If $a(v_j) \neq 0$, then the following row and column operations annihilate the entries kj and jk:

$$R_k \leftarrow R_k - \frac{\omega_{jk}}{a(v_j)} R_j \quad \text{and} \quad C_k \leftarrow C_k - \frac{\omega_{jk}}{a(v_j)} C_j.$$

After these two operations, the corresponding entries of the matrix are

$$\begin{array}{c} \\ k \\ \\ j \\ \\ \end{array}\begin{bmatrix} & \vdots & & \vdots & \\ \cdots & a(v_k) - \frac{(\omega_{kj})^2}{a(v_j)} & \ldots & 0 & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & 0 & \cdots & a(v_j) & \cdots \\ & \vdots & & \vdots & \end{bmatrix}.$$

Note that if v_k has all children with nonzero diagonal values, each of them may be used to annihilate the two off-diagonal entries that correspond to its connection with v_k. Hence, after performing the same operations for all children of v_k, the diagonal value of v_k becomes

$$a(v_k) - \sum_{v_i \in C_k} \frac{(\omega_{ik})^2}{a(v_i)},$$

which corresponds to the value assigned by Algorithm 1 in this case (Step 1).

Suppose that v_k has a child v_j with $a(v_j) = 0$, as in the submatrix below. Then vertex v_j may be used to annihilate the two off-diagonal entries of any other child v_i of v_k, as well as the two entries representing the edge between v_k and its parent v_ℓ, in the case v_k is not the root, as follows. Note that at this point v_k and v_ℓ still have their initial diagonal values, since the vertices are processed bottom-up.

$$\begin{array}{c} \ell \\ k \\ j \\ i \end{array}\begin{bmatrix} \delta_\ell + \alpha & \omega_{\ell k} & & \\ \omega_{k\ell} & \delta_k + \alpha & \omega_{kj} & \omega_{ki} \\ & \omega_{jk} & 0 & \\ & \omega_{ik} & & a(v_i) \end{bmatrix}.$$

The operations

$$R_i \leftarrow R_i - \frac{\omega_{ik}}{\omega_{jk}} R_j \quad \text{and} \quad C_i \leftarrow C_i - \frac{\omega_{ki}}{\omega_{kj}} C_j,$$

annihilate the entries ik and ki, while the operations below annihilate the entries ℓk and $k\ell$:

$$R_\ell \leftarrow R_\ell - \frac{\omega_{\ell k}}{\omega_{jk}} R_j \quad \text{and} \quad C_\ell \leftarrow C_\ell - \frac{\omega_{k\ell}}{\omega_{kj}} C_j.$$

Note that $\omega_{jk} = \omega_{kj} \neq 0$, since v_j is a child of v_k.

These last two operations effectively remove the edge between v_k and its parent v_ℓ, disconnecting the graph, which is performed in Step 2 of Algorithm 1. At this point, the submatrix with rows and columns i, j, k, ℓ has been transformed as

$$\begin{array}{c} \ell \\ k \\ j \\ i \end{array} \begin{bmatrix} \delta_\ell + \alpha & 0 & & \\ 0 & \delta_k + \alpha & \omega_{kj} & 0 \\ & \omega_{jk} & 0 & \\ & 0 & & a(v_i) \end{bmatrix}.$$

Next, the operations

$$R_k \leftarrow R_k - \frac{(\delta_k + \alpha)}{2\omega_{jk}} R_j \quad \text{and} \quad C_k \leftarrow C_k - \frac{(\delta_k + \alpha)}{2\omega_{kj}} C_j$$

annihilate the entry kk and the submatrix becomes

$$\begin{array}{c} \ell \\ k \\ j \\ i \end{array} \begin{bmatrix} \delta_\ell + \alpha & 0 & & \\ 0 & 0 & \omega_{kj} & 0 \\ & \omega_{jk} & 0 & \\ & 0 & & a(v_i) \end{bmatrix}.$$

Finally, the operations

$$\begin{aligned} R_j &\leftarrow R_j + \tfrac{1}{\omega_{kj}} R_k, & C_j &\leftarrow C_j + \tfrac{1}{\omega_{jk}} C_k, \\ R_k &\leftarrow R_k - \tfrac{\omega_{kj}}{2} R_j, & C_k &\leftarrow C_k - \tfrac{\omega_{jk}}{2} C_j \end{aligned}$$

yield the diagonalized form

$$\begin{array}{c} \ell \\ k \\ j \\ i \end{array} \begin{bmatrix} \delta_\ell + \alpha & 0 & & \\ 0 & -\frac{(\omega_{kj})^2}{2} & 0 & 0 \\ & 0 & 2 & \\ & 0 & & a(v_i) \end{bmatrix}.$$

The diagonal values of v_j and v_k obtained after the operations above are exactly the values directly assigned by the algorithm in Step 2. We also note that all other children of v_k are unaffected by the operations above, including those that might have zero values.

Considering that the diagonal values computed by algorithm *DiagonalizeW*(T, α) were obtained by elementary row and column operations, such that each operation performed in one row was also performed in the corresponding column, it follows that the diagonal matrix D_α, whose entries are the diagonal values assigned by the algorithm, is congruent to ${}^{D}\!\mathcal{L}(T) + \alpha I$. Therefore, by Sylvester's Law of Inertia (see [6, Theorem 4.5.8]), we have the next result.

Theorem 2.1. *Given a real diagonal matrix D, let D_α be the diagonal matrix produced by the algorithm DiagonalizeW$(T, -\alpha)$ for a weighted tree T and real number α. Then, the number of positive, negative and zero diagonal entries in D_α is equal to the number of eigenvalues of the perturbed Laplacian matrix of T with respect to D, ${}^{D}\!\mathcal{L}(T)$, which are greater, smaller and equal to α, respectively.*

Example 2.2. Let T be the weighted tree with vertices v_1, v_2, v_3, v_4 and v_5 on the left side of Figure 1, with weights represented on the edges. Suppose that the diagonal entries of matrix D are $\delta_1 = 1,\ \delta_2 = 2,\ \delta_3 = 1,\ \delta_4 = 1$ and $\delta_5 = 1$.

Let us apply algorithm *DiagonalizeW*(T, α), with $\alpha = -2$. The initial diagonal values assigned to the vertices are $a(v_i) = \delta_i - 2$, for $i = 1, \ldots, 5$, which are represented on the vertices of T, on the left side of Figure 1.

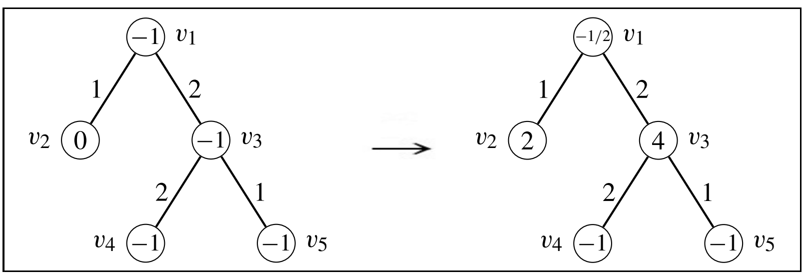

Figure 1: Algorithm *DiagonalizeW*(T, α) with $\alpha = -2$.

Vertices v_4 and v_5 are the children of v_3 and have nonzero values, hence the algorithm assigns

$$a(v_3) = -1 - \frac{(\omega_{34})^2}{a(v_4)} - \frac{(\omega_{35})^2}{a(v_5)} = -1 - \frac{2^2}{-1} - \frac{1^2}{-1} = 4.$$

Vertex v_1 has a child with a zero value (vertex v_2), then the algorithm assigns value 2 to v_2, whereas the diagonal value of v_1 becomes

$$a(v_1) = -\frac{(\omega_{12})^2}{2} = -\frac{1^2}{2} = -\frac{1}{2}.$$

Therefore, since the algorithm produced two positive and three negative diagonal values, it follows from Theorem 2.1 that the matrix ${}^{D}\!\mathcal{L}(T)$ has two eigenvalues greater than 2 and three eigenvalues smaller than 2. Applying the algorithm with $\alpha = 0$, we obtain that ${}^{D}\!\mathcal{L}(T)$ has one negative and four positive eigenvalues, while with $\alpha = -1$ we get that $\mathcal{L}(T)$ has one eigenvalue equal to 1, two eigenvalues smaller than 1 and two eigenvalues greater than 1. Hence, $\mathcal{L}(T)$ has one negative eigenvalue, one eigenvalue in the interval (0, 1), one eigenvalue equal to 1 and two eigenvalues greater than 2.

3 TREES WITH AT MOST FIVE DISTINCT EIGENVALUES WITH RESPECT A FAMILY OF PERTURBED LAPLACIAN MATRICES

In this section we apply Algorithm *DiagonalizeW*(T, α) to study trees that have up to 5 distinct perturbed Laplacian eigenvalues.

It follows from the Theorem below, whose proof can be found in [2, Proposition 1.3.3], that we only need to consider trees with a small diameter. We recall that the *diameter* of a graph is the maximum distance between any two vertices in the graph.

Theorem 3.1. *If G is a connected graph with diameter d and $M = (m_{ij})$ is a nonnegative symmetric matrix with rows and columns indexed by the vertices of G and such that for distinct vertices v_i, v_j we have $m_{ij} > 0$ if and only if v_i and v_j are adjacent, then G has at least $d+1$ distinct eigenvalues with respect to M.*

We show next that this property is more general. For that matter we apply a result due to Schur, whose proof can be found in [6, Theorem 4.3.45].

Theorem 3.2 (Schur). *Let A be a real symmetric matrix of order n with diagonal entries $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$ and eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. Then*

$$\sum_{i=1}^{k} d_i \leqslant \sum_{i=1}^{k} \lambda_i, \quad \textit{for } k = 1, \ldots, n-1, \quad \textit{and} \quad \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i .$$

Theorem 3.3. *If G is a connected graph with positive weights and diameter d, then any perturbed Laplacian matrix of G has at least $d+1$ distinct eigenvalues.*

Proof. Let G be a connected graph with order n, positive weights and diameter d. Let $\mathcal{L}(G) = D - A$ be the perturbed Laplacian matrix of G with respect to a diagonal matrix $D = (d_{ij})$. Let us denote the entries of the adjacency matrix of G by a_{ij}, with $i, j \in \{1, \ldots, n\}$. Let

$$m = 1 + \max_{1 \leqslant k \leqslant n} \{\lambda_k\},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathcal{L}(G)$, and consider the matrix $B = mI - \mathcal{L}(G) = mI - D + A$. Then, for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, $b_{ij} = a_{ij} \geqslant 0$. Besides, for all $i \in \{1, \ldots, n\}$, the diagonal entry $b_{ii} = m - d_{ii}$ of B is positive. This follows from the fact that $\mathcal{L}(G)$ is symmetric and then, by Theorem 3.2, $d_{ii} \leqslant \max_{1 \leqslant k \leqslant n} \{\lambda_k\} < m$. Therefore, by Theorem 3.1, B has at least $d+1$ distinct eigenvalues. Since the eigenvalues of B are $m - \lambda_1, \ldots, m - \lambda_n$, it follows that $\mathcal{L}(G)$ has at least $d+1$ distinct eigenvalues. $\square$

The perturbed Laplacian matrix of a graph depends on an arbitrary diagonal matrix D. We consider the case where $D = \mu I$, for some $\mu \in \mathbb{R}$, thus the perturbed Laplacian matrix of a graph G is of the form

$$\mathcal{L}^{\mathcal{P}}(G) = \mu I - A.$$

Note that, in particular, if $\mu = 0$, $\mathcal{L}^{\mathcal{P}}(G) = -A$. Besides, if G has no isolated vertices, $\mu = 1$ and the weights of G are

$$\omega_{ij} = \frac{1}{\sqrt{d_i d_j}},$$

then $\mathcal{L}^{\mathcal{P}}(G)$ is the normalized Laplacian matrix of G', where G' is an unweighted graph with the same edges and vertices then G. Hence the adjacency matrix of a weighted graph and the normalized Laplacian matrix of an unweighted graph with no isolated vertices are special cases of a perturbed Laplacian matrix of the form $\mathcal{L}^{\mathcal{P}}(G) = \mu I - A$.

It is known that the spectrum of the adjacency matrix of a connected graph G is symmetric if and only if G is bipartite (see [2, Proposition 3.4.1]). The family of perturbed Laplacian matrices that we are considering satisfies a similar result. It is easy to see that the spectrum of $\mu I - A$ is symmetric about $\mu \in \mathbb{R}$ if and only if the spectrum of A is symmetric. In fact, if λ is an eigenvalue of $\mu I - A$, then $\mu - \lambda$ is an eigenvalue of A. Hence, if the spectrum of A is symmetric, then $\lambda - \mu$ is also an eigenvalue of A. Thus, $\mu - (\lambda - \mu) = 2\mu - \lambda$ is an eigenvalue of $\mu I - A$. The converse is similar.

Theorem 3.4. *Let G be a connected weighted graph of order n and $\mu \in \mathbb{R}$. Then G is bipartite if and only if the spectrum of $\mathcal{L}^{\mathcal{P}}(G) = \mu I - A$ is symmetric about μ.*

In order to characterize the trees that have at most five distinct eigenvalues for perturbed Laplacian matrices of the form $\mu I - A$, for some $\mu \in \mathbb{R}$, by Theorem 3.3 it is enough to consider trees with diameter smaller than five, since every tree is a connected graph. Besides, every tree T is a bipartite graph, so the spectrum of $\mathcal{L}^{\mathcal{P}}(T) = \mu I - A$ is symmetric about μ.

Let T be a tree with n vertices and diameter d less than or equal to 4. If $d = 1$, T is the complete graph with two vertices and has two different eigenvalues: $\mu + \omega$ and $\mu - \omega$, where ω is the weight of the edge that connects the vertices.

In the case $d = 2$, T is the star S_n, that has exactly three distinct eigenvalues, symmetric about μ, which is an eigenvalue with multiplicity to $n - 2$. To see that, we apply algorithm *DiagonalizeW*(S_n, α), with $\alpha = -\mu$ and the vertex with degree $n - 1$ as the root. Since all $n - 1$ pendants of S_n receive zero diagonal values, the algorithm assigns to the root v the diagonal value $-\frac{(\omega_{vy^*})^2}{2}$, where y^* is a pendant of v selected to receive value 2. Therefore, exactly $n-2$ vertices have a zero diagonal value at the end of the execution, which implies that μ is an eigenvalue with multiplicity to $n - 2$, by Theorem 2.1.

Now we consider the case $d = 3$. Note that any diameter 3 tree can be seen as two stars S_{k+1} and $S_{\ell+1}$, where $k, \ell \geqslant 1$, with an edge linking their centers, as illustrated in Figure 2.

Applying algorithm *DiagonalizeW*, we obtain the following result.

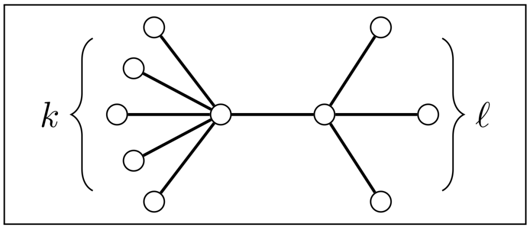

Figure 2: A diameter 3 tree.

Theorem 3.5. *Let T be a diameter 3 tree. Then all eigenvalues of $\mathcal{L}(T) = \mu I - A$, except possibly for μ, are simple. Moreover, $\mathcal{L}(T)$ has exactly four eigenvalues if and only if $T = P_4$, the path with four vertices. Otherwise, $\mathcal{L}(T)$ has exactly five distinct eigenvalues.*

Proof. Let T be a diameter 3 tree with $n = k + \ell + 2$ vertices, as shown in Figure 2. By Theorem 3.3, $\mathcal{L}(T) = \mu I - A$ has at least four distinct eigenvalues. If $n = 4$, T is the path P_4 and $\mathcal{L}(T)$ has exactly four distinct eigenvalues. Suppose that $n > 4$ and let us apply the algorithm *DiagonalizeW*(T, α), with $\alpha = -\mu$ and T rooted at the vertex with ℓ pendants. Then all vertices are initialized with a zero value. Since all k pendants of the vertex adjacent to the root have zero values, when this vertex is processed its diagonal value becomes negative and one of its k pendants receives a positive value. Besides, since $\ell \geqslant 1$, the root has at least one pendant with a zero value. Hence, when the algorithm processes the root, its diagonal value becomes negative and exactly one of its ℓ pendants receives a positive value. Therefore, at the end of the execution we obtain $k + \ell - 2 = n - 4$ zero values. By Theorem 2.1, it follows that μ is an eigenvalue of $\mathcal{L}(T)$ with multiplicity $n - 4$. Then the other four eigenvalues of $\mathcal{L}(T)$ must be distinct since it has at least four distinct eigenvalues and its spectrum is symmetric about μ. Hence, $\mathcal{L}(T)$ has exactly five distinct eigenvalues if $n > 4$, which concludes the proof. $\square$

If T has diameter $d = 4$, then $\mathcal{L}(T)$ has at least five distinct eigenvalues. Hence, the path P_4 is the only tree with exactly four distinct eigenvalues. We want to characterize the trees, if any, necessarily of diameter 4, that have exactly five distinct eigenvalues for the perturbed Laplacian matrix $\mu I - A$. Every diameter 4 tree is of the form depicted in Figure 3: it contains a vertex v adjacent to $k \geqslant 2$ vertices $v_1, \ldots, v_k$, where v_i has degree $p_i + 1$, with $p_i \geqslant 1$, for $i = 1, \ldots, k$, and each v_i is adjacent to p_i pendants, and v is also possibly adjacent to $m \geqslant 0$ pendants. In this case, we write $T = \mathcal{T}(k, p_1, p_2, \ldots, p_k, m)$.

The following result gives the multiplicity of μ as an eigenvalue of $\mathcal{L}(T) = \mu I - A$, where T is a diameter 4 tree.

Theorem 3.6. *For any diameter 4 tree of the form $T = \mathcal{T}(k, p_1, p_2, \ldots, p_k, m)$, the multiplicity of μ as an eigenvalue of $\mu I - A$ is $1 - k + \sum_{i=1}^{k} p_i \geqslant 1$, when $m = 0$, and $m - 1 - k + \sum_{i=1}^{k} p_i$, when $m > 0$.*

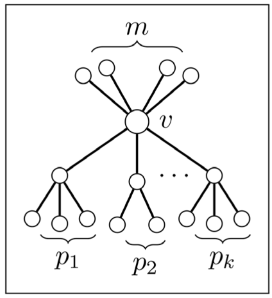

Figure 3: A diameter 4 tree $\mathcal{T}(k, p_1, p_2, \ldots, p_k, m)$.

Proof. Let us apply the algorithm *DiagonalizeW*(T, α), with $\alpha = -\mu$, to T rooted at vertex v of degree $k+m$, which is adjacent to the vertices $v_1, v_2, \ldots, v_k$ of degree $p_1+1, \ldots, p_k+1$, respectively.

Suppose that $m = 0$, that is, v has no pendants. Since $\mu+\alpha = 0$, initially all vertices are assigned a zero diagonal value, as the left hand-side of Figure 4 shows. Next, for $i = 1, \ldots, k$, the value of v_i becomes $-\frac{(\omega_{v_i v_i^*})^2}{2}$, where v_i^* is a pendant of v_i chosen to take value 2, and the edge connecting v_i to v is removed, so that the value of the root v remains 0. The other $\sum(p_i-1)$ pendants also remain with a zero diagonal value, as illustrated in the right-hand side of Figure 4. Therefore, by Theorem 2.1, the multiplicity of μ as an eigenvalue of $\mathcal{L}(T) = \mu I - A$ is exactly

$$\sum_{i=1}^{k}(p_i-1)+1 = \left(\sum_{i=1}^{k} p_i\right) - k + 1 \geqslant 1.$$

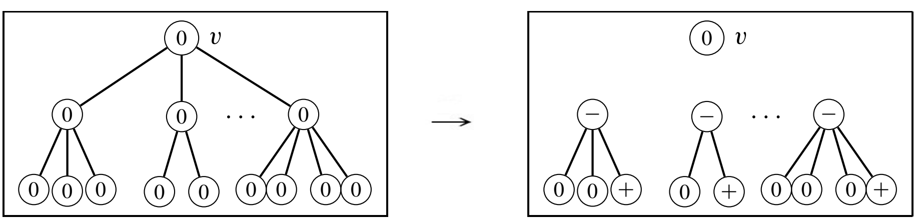

Figure 4: A diameter 4 tree with $m = 0$.

Now suppose that $m > 0$. Figure 5 illustrates the execution the algorithm in this case. All the edges connecting the v_i's to v are also removed.

When the root v is processed, since it remains connected only to its m pendants, which have value 0, the value assigned to v becomes $-\frac{(\omega_{vv^*})^2}{2}$, where v^* is a pendant of v chosen to take value 2. Hence, by Theorem 2.1, the multiplicity of μ as an eigenvalue of $\mathcal{L}(T) = \mu I - A$ is

$$\left(\sum_{i=1}^{k} p_i\right) - k + (m-1). \qquad \square$$

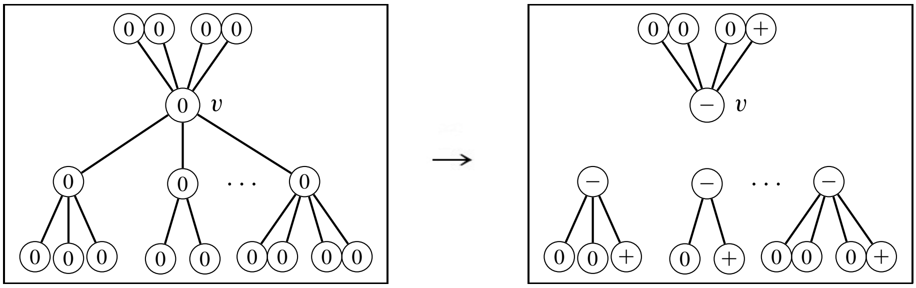

Figure 5: A diameter 4 tree with $m > 0$.

The next result gives the multiplicity of the eigenvalues of $\mathcal{L}(T) = \mu I - A$ that are different from μ for a diameter 4 tree T. For $i = 1, \ldots, k$, let ω_{ih} be the weight of the edge that connects v_i to its pendant q_{ih}, for $h = 1, \ldots, p_i$, and let $\sigma_i = \sum_{h=1}^{p_i} (\omega_{ih})^2$.

Theorem 3.7. *Let $T = \mathcal{T}(k, p_1, p_2, \ldots, p_k, m)$ be a diameter 4 tree. Then $\mathcal{L}(T) = \mu I - A$ has an eigenvalue different from μ with multiplicity $t \geqslant 2$ if and only if $t < k$ and exactly $t + 1$ σ_i's are equal. Moreover, $\lambda_1 = \mu - \sqrt{\sigma}$ and $\lambda_2 = \mu + \sqrt{\sigma}$ are eigenvalues of $\mathcal{L}(T) = \mu I - A$ with multiplicity $t \geqslant 1$ if $\sigma_i = \sigma$ for exactly $t + 1$ v_i's.*

Proof. Let us suppose that $\lambda \neq \mu$ is an eigenvalue of $\mathcal{L}(T)$ with multiplicity $t \geqslant 2$. Applying algorithm *DiagonalizeW*$(T, -\lambda)$ to T rooted at vertex v of degree $k + m$, each pendant of T receives the initial diagonal value $\mu - \lambda \neq 0$. Hence,

$$a(v_i) = \mu - \lambda - \sum_{h=1}^{p_i} \frac{(\omega_{ih})^2}{\mu - \lambda} = \mu - \lambda - \frac{\sigma_i}{\mu - \lambda},$$

for all $1 \leqslant i \leqslant k$. Due to the multiplicity of λ, the algorithm produces exactly t zero diagonal values. Considering that each pendant of T has a nonzero diagonal value and $t \geqslant 2$, then $a(v_i) = 0$ for at least one i, since otherwise the only possible zero diagonal value would be $a(v)$, which contradicts the fact that $t \geqslant 2$. Thus, the only way to obtain exactly t zero diagonal values at the end of the algorithm is that $a(v_i) = 0$ for exactly $t + 1$ v_i's, so that, after processing vertex v, the diagonal value of v is negative and one of those $t + 1$ v_i's has a positive diagonal value. This implies that $t + 1 \leqslant k$. Besides, for $1 \leqslant i < j \leqslant k$,

$$a(v_i) = a(v_j) \Leftrightarrow \mu - \lambda - \frac{\sigma_i}{\mu - \lambda} = \mu - \lambda - \frac{\sigma_j}{\mu - \lambda} \Leftrightarrow \sigma_i = \sigma_j.$$

Without loss of generality, now let us suppose that for some t, $1 \leqslant t < k$, there exists $\sigma \in \mathbb{R}$ such that $\sigma_i = \sigma$, for all i, $1 \leqslant i \leqslant t + 1$, and $\sigma_i \neq \sigma$, for $i > t + 1$. We apply algorithm *DiagonalizeW*(T, α) to T rooted at vertex v of degree $k + m$ and $\alpha = -\lambda$, where $\lambda = \mu - \sqrt{\sigma}$. Initially all pendants of T are assigned a diagonal value $\mu - \lambda = \sqrt{\sigma}$, which is positive. Besides, for each i, $1 \leqslant i \leqslant t + 1$, v_i is assigned a zero value, since

$$a(v_i) = \mu - \lambda - \frac{\sigma_i}{\mu - \lambda} = \frac{(\mu - \lambda)^2 - \sigma}{\mu - \lambda} = 0.$$

Therefore, when vertex v is processed, we obtain $a(v_j) = 2$, for exactly one j in $\{1, \ldots, t+1\}$ and $a(v) = -\frac{(\omega_{vv_j})^2}{2} < 0$. Hence, at the end of the execution, there are exactly t zero diagonal values, which implies that λ is an eigenvalue of $\mathcal{L}^{P}(T)$ with multiplicity t. The result for $\lambda = \mu + \sqrt{\sigma}$ follows from the symmetry of the spectrum of $\mathcal{L}^{P}(T) = \mu I - A$ with respect to μ (Theorem 3.4). □

Corollary 3.1. *If $\sigma_i \neq \sigma_j$ for all $1 \leqslant i < j \leqslant k$, then, except possibly by μ, all eigenvalues of $\mathcal{L}^{P}(T) = \mu I - A$ are simple.*

The result below characterizes the diameter 4 trees for which the perturbed Laplacian matrix of the form $\mu I - A$ has exactly 5 distinct eigenvalues.

Theorem 3.8. *Let $T = \mathcal{T}(k, p_1, p_2, \ldots, p_k, m)$ be a diameter 4 tree. If $m = 0$ and $k = 2$, or $m = 0$, $k \geqslant 3$ and $\sigma_i = \sigma_j$, for all $1 \leqslant i < j \leqslant k$, then $\mathcal{L}^{P}(T) = \mu I - A$ has exactly 5 distinct eigenvalues. Otherwise, $\mathcal{L}^{P}(T)$ has at least 6 distinct eigenvalues.*

Proof. If $m = 0$, then, by Theorem 3.6, the multiplicity of μ as an eigenvalue of $\mathcal{L}^{P}(T) = \mu I - A$ is $1 - k + \sum_{i=1}^{k} p_i \geqslant 1$. Hence, $\mathcal{L}^{P}(T)$ has exactly $2k$ eigenvalues different from μ. If $k = 2$, the result is clear, since T has at least 5 distinct eigenvalues by Theorem 3.3. Let us suppose that $k \geqslant 3$. If $\sigma_i = \sum_{h=1}^{p_i} (\omega_{ih})^2 = \sigma$, for all $1 \leqslant i \leqslant k$, by Theorem 3.7, $\lambda_1 = \mu + \sqrt{\sigma}$ and $\lambda_2 = \mu - \sqrt{\sigma}$ are eigenvalues of $\mathcal{L}^{P}(T)$ with multiplicity $k - 1 \geqslant 1$. Hence $\mathcal{L}^{P}(T)$ has $2k - (2(k-1)) = 2$ eigenvalues different from μ, λ_1 and λ_2. These two eigenvalues are simple, since the spectrum of $\mathcal{L}^{P}(T)$ is symmetric about μ (Theorem 3.4), which shows that $\mathcal{L}^{P}(T)$ has exactly 5 distinct eigenvalues.

However, if $\sigma_i = \sigma$, for all $1 \leqslant i \leqslant t$, for some $2 \leqslant t < k$, and $\sigma_i \neq \sigma$, for all $i > t$, the multiplicity of λ_1 and λ_2 is $t - 1$. Hence $\mathcal{L}^{P}(T)$ has $2(k - t + 1) \geqslant 4$ eigenvalues different from μ, λ_1 and λ_2, which implies that $\mathcal{L}^{P}(T)$ has at least 7 distinct eigenvalues. If $\sigma_i \neq \sigma_j$, for all $1 \leqslant i < j \leqslant k$, by Corollary 3.1, $\mathcal{L}^{P}(T)$ has $2k \geqslant 6$ simple eigenvalues different from μ, which shows that $\mathcal{L}^{P}(T)$ has at least 7 distinct eigenvalues. Finally, if $m > 0$ and $k \geqslant 2$, by Theorem 3.6, the multiplicity of μ is $m - 1 - k + \sum_{i=1}^{k} p_i \geqslant 0$ and $\mathcal{L}^{P}(T)$ has $2k + 2 \geqslant 6$ eigenvalues different from μ. By Theorems 3.4 and 3.7, it follows that $\mathcal{L}^{P}(T)$ has at least 6 distinct eigenvalues, since in this case the multiplicity of μ can be zero. □

RESUMO. Nós apresentamos um algoritmo de tempo linear para calcular o número de autovalores de uma matriz laplaciana perturbada qualquer associada a uma árvore, num dado intervalo real. Este algoritmo pode ser aplicado a árvores com ou sem pesos. Utilizando este procedimento, obtemos uma caracterização das árvores com até cinco autovalores distintos para uma família de matrizes laplacianas perturbadas, que inclui a matriz de adjacências e a matriz laplaciana normalizada como casos particulares, entre outras.

Palavras-chave: matriz laplaciana perturbada, localização de autovalores, árvores.

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